CHAOS, INVERSE LIMIT SPACES AND RECURRENCE

By

SHIHAI LI

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By

Shihai Li

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In 1975, Li and Yorke gave a definition of chaos on a compact metric space. On the interval, if a continuous map has positive topological entropy, then it is chaotic. But the converse is not true. In this thesis, we define a concept called ω-chaos. The definition of ω-chaos parallels the definition of chaos given by Li and Yorke. But a continuous map is ω-chaotic if and only if it has positive topological entropy. Moreover, we prove that a continuous map is ω-chaotic if and only if it is chaotic in the sense of Devaney.

If a continuous map f from a compact metric space to itself is ω-chaotic, then the shift map σ_f on the inverse limit space induced by it is also ω-chaotic. With some modification of the definition, the converse is true. σ_f is chaotic in the sense of Devaney if and only if f is chaotic in the sense of Devaney. An expanding map under which every point is nonwandering on a 1-branched manifold is both ω-chaotic and chaotic in the sense of Devaney. We give a sufficient condition for a transitive continuous map to have sensitive dependence on initial conditions. A new method of

constructing chaotic maps on the pseudoarc is given. On any compact metric space, we prove that the chain recurrent set of the shift map equals the inverse limit space of the chain recurrent set of the sole bonding map. Similar results are proved for the nonwandering set, ω-limit set, recurrent set, and almost periodic set.

On the interval, we also prove that a nonwandering point, which is not an ω-limit point, is in the orbit of a turning point. We say a point is a γ-limit point if it is both an ω- limit point and an α- limit point of a point. A nonwandering point which is not a γ-limit is in the closure of the orbit of turning points.

At last we study Type 2^∞ maps on the interval and the structure of Type 2^∞ unimodal maps.

CHAPTER 1

INTRODUCTION

In this thesis, we study the complexity of topological dynamical systems on compact metric spaces. The main body of the thesis has four chapters. In the next two chapters we mainly study chaos. The fourth chapter is about the relation between turning points and chain recurrent points. The fifth chapter is about Type 2^∞ maps and unimodal maps. The sixth chapter is about further research.

Chaotic behaviors have been observed and studied in various fields. Mathematicians and scientists are paying more and more attention to this subject. The concept of chaos in science was first introduced in themodynamics. Classical thermodynamics was constructed with the awareness of molecular chaos in the background. The concept of modern chaos was born with Lorenz's attractor in 1961. The first mathematical definition was given by Li and Yorke [LiY]. Since then various definitions have been given, e.g. positive entropy, the definition of Devaney [Dev], topological mixing, the definition of Xiong [Xio, IC/88/385] and so on. In applications, for the convenience of study, people give definitions of chaos which may not be equivalent to others. But fortunately the definitions listed above are related for continuous maps on the interval.

The concept of entropy was invented by Clausius in 1854. Shannon carried it over to information theory in 1948, and Kolmogorov to ergodic theory in 1958. In each setting entropy is a measure of randomness or disorder. Adler, Konheim and McAndrew introduced topological entropy in 1965 [AKM]. Later, Bowen [Bow, 1971] gave a different definition for (possibly noncompact) metric spaces. Topological entropy has relations with metric entropy. Goodwyn [Gow, 1969, 1972], Dinaburg

[Din, 1970], and Goodman [Gom, 1971] have proved that the topological entropy of a homeomorphism is the supremum of its metric entropies over all ergodic invariant Borel probability measures. In this thesis, for our convenience, we use "entropy" for "topological entropy" sometimes.

A great deal of work has been done concerning the relation between positive entropy and the definiton of chaos given by Li-Yorke. We summarize this work as follows.

Since the study of chaos involves the Shărkovskii Theorem, we will introduce the theorem first. A point x is called a periodic point of a continuous map f defined on a compact metric space X if there is an integer $n > 0$ such that $f^n(x) = x$. The smallest such n is called the period of x. If $n = 1$ then we say that x is a fixed point. The collection of x, $f(x)$, ..., $f^{n-1}(x)$ is called the periodic orbit of x. The following order of natural integers is called the Shărkovskii Order.

$$3 \rhd 5 \rhd 7 \rhd 9 \rhd \ldots \rhd$$

$$2 \cdot 3 \rhd 2 \cdot 5 \rhd 2 \cdot 7 \rhd 2 \cdot 9 \rhd \ldots \rhd$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots \rhd 2^n \rhd \ldots \rhd 4 \rhd 2 \rhd 1.$$

The Shărkovskii Theorem [Shar, 1964a] says that if a continuous map on the interval has a periodic point of period m, then it must have periodic points of periods n for $m \rhd n$. Simpler proofs of the Shărkovskii Theorem were given by Block, Guckenheimer, Misiurewicz and Young [BloGMY], and Ho and Morris [HoM] independently. The existence of periodic points of period $m2^n$ with $m > 1$ an odd number characterize continuous maps with positive entropy. Block has proved in [Blo, 1978a] that if f has zero entropy then f is a map in $C(2^\infty)$, which is the collection of continuous maps with no period of the form $m2^n$ ($m > 1$ odd), and Misiurewicz [Mis] has proved the converse, that is, if f has positive entropy, then f is not in $C(2^\infty)$.

Li and Yorke [LiY] proved that, on the interval, if f has a periodic point with period three, then f is chaotic. Later in 1977, Nathanson [Nat] proved that period five and period seven implies chaos. In 1978, Oono [Oo] generalized the result and proved that if f has a periodic point with period not a power of two, then f is chaotic. As we have pointed out in the last paragragh, a continuous map f of the interval has positive topological entropy if and only if $f \notin C(2^\infty)$. If a continuous map has positive topological entropy, then it is chaotic according to the definition of Li and Yorke.

But the converse of this result is false. Xiong [Xio, 1986a] and Smitál [Smi] have given counterexamples independently. Then one can ask the following question. Can this definition be modifided so that the converse is true? In chapter 2, we will prove the following theorem which answers this question.

THEOREM 2.4.8. *Let f be a continuous map of a compact interval I to itself. The following statements are equivalent.*

I) f has positive topological entropy.

II) There is an uncountable ω-scrambled set S of I such that

$$\bigcap_{x \in S} \omega(x, f) \neq \phi.$$

III) f is ω-chaotic.

IV) There is an ω-scrambled set with two points.

The definition of ω-chaos and ω-scrambled set are given in section 2.1. They are similar to the definitions of chaos and scrambled set given by Li and Yorke.

Devaney gave a definition of chaos in 1986 [Dev, 1986]. He defines chaotic maps as the maps possessing transitivity, sensitive dependence on initial dependence (we abbreviate it as SDIC) and dense periodic points. We will prove the following theorem in section 2.5.

THEOREM 2.5.6. *Let f be a continuous map of a closed interval I to itself. The following statements are equivalent.*

I) f has positive topological entropy.

V) There is a closed invariant subset D of I which is chaotic in the sense of Devaney.

VI) There is a set D as in V) and an uncountable ω-scrambled set $S \subset D$.

A continuous map on the interval is topologically mixing if and only if f^n is transitive for all $n > 0$ as shown by Barge and Martin [BarM, 1987]. Since a transitive map on the interval has entropy at least $\log 2$, it follows that a topologically mixing map on the interval has positive entropy [BarM, 1987]. Xiong [Xio, IC/88/385] has recently proved that *a continuous map on the interval has positive entropy if and only if there is a closed, invariant set such that the restriction of f on the subset satisfies the following conditions.*

1) The periodic points are dense.

2) There are infinitely many periods.

3) The map is topologically mixing.

REMARK. *If this closed, invariant set is an interval, then it must be a chaotic set in the sense of Devaney by Theorem 2.5.6.*

Some attractors can be characterized as the inverse limit spaces of simpler spaces, such as intervals and branched manifolds. Is it enough to study the chaotic behavior on a simpler space in order to study the chaotic behavior on an attractor? The answer in this case is YES. In section 3.2 we will prove the following theorems.

THEOREM 3.2.1. *Let X be a compact metric space, and $f : X \to X$ be a continuous map. If f is $\omega-$chaotic, then so is $\sigma_f : \varprojlim(X, f) \to \varprojlim(X, f)$.*

In Theorem 3.2.1, σ_f denotes the shift map on the inverse limit space.

THEOREM 3.2.4. *Let X be a compact metric space, and $f : X \to X$ be a continuous map. σ_f is chaotic in the sense of Devaney iff f is. Furthermore, assuming D is a closed, invariant set. D is a chaotic set if and only if $\varprojlim(D, f)$ is a chaotic set.*

It is known that if a continuous onto map f on the interval I is topologically transitive, then the periodic points of f are dense in I. This follows from the fact that $\overline{P(f)} = \Omega(f|_\Omega)$ [Xio, 1983b], [Blo], [Nit, 1980]. Barge and Martin have proved [BarM, 1985] that if f is transitive, then every point of $y \in I$ is *unstable* in the sense that there is a number $\delta = \delta(y) > 0$, such that for any neighborhood N of y, there is a point $z \in Z$ and an integer $n > 0$ such that $|f^n(y) - f^n(z)| > \delta$. We will show in section 2.4 that there is a $\delta > 0$ which does not depend on y. Using this we can prove that if a continuous map on a closed invariant subset of the interval with nonempty interior is transitive then it has SDIC. Thus in the case a transitive map is chaotic in the sense of Devaney on the whole set, we obtain the following theorem.

THEOREM 3.4.6. *Suppose that $D \subset I$ is a closed, invariant subset containing an interval. If $f : I \to I$ is transitive on D, then both f and σ_f are chaotic in the sense of Devaney with chaotic sets D and $\varprojlim(D, f)$, respectively, and both f and σ_f are $\omega-$chaotic.*

In this thesis we will also study the chain recurrent points, nonwandering points, ω-limit points, recurrent points and periodic points. The structure of these sets reflects the complexity of a dynamical system. Bowen [Bow] has proved that the topological entropy of a dynamical system is the same as the topological entropy of the subsystem restricted to the nonwandering set. In the definition given by Devaney, every point in the chaotic set is in the closure of periodic points. In the definition of ω-chaos, we use ω-limit points to characterize chaos.

The study of recurrence begins with the study of fixed points. In $\mathbf{R}^n$, let $D \subset \mathbf{R}^n$ be a compact ball. Brouwer's Fixed Point Theorem says that a continuous self-map f on D has a fixed point on D. On a general compact set in $\mathbf{R}^n$ the theorem

may fail. For example, an irrational rotation on the circle has no fixed point. A closed and invariant subset of a topological space, which has no proper nonempty closed and invariant subset is called a minimal set. Obviously every periodic orbit is a minimal set. It is known that if M is a closed and invariant subset on a compact metric space, then M contains a minimal set. One can use Zorn's Lemma to prove the fact. A point x is called an *almost periodic point* if, for any neighborhood U of x, $\{n \geq 0;\ f^n(s) \in U\}$ has bounded gaps, i.e. if we write $\{n \geq 0;\ f^n(s) \in U\}$ as $\{0 = n_0 < n_1 < n_2 < \dots\}$, then $\{n_{k+1} - n_k\}$ is bounded. A classical result proved by Birkhoff [Bir, 1927], [GotH, 1955] shows that if M is a minimal set, then every point in M is almost periodic. Furthermore if x is almost periodic, then $\overline{Orb(x)}$ is a minimal set, where $Orb(x) = \{y : y = f^n(x) \text{ for } n \geq 0\}$ is the orbit of x under f. Denote the collection of almost periodic points by $AP(f)$ or AP. Then AP is the union of all minimal sets. If M is an infinite minimal set then it is uncountable.

A point is called a *recurrent point* if it is an ω-limit point of itself. The collection of recurrent points is denoted by $R(f)$ or R. Then $P \subset AP \subset R$, where P is the collection of all periodic points of f. Every almost periodic point is recurrent. But the converse is not true. Here is an example. Let $\Sigma_2 = \{0,\ 1\}^{\ N}$ with the topology defined by the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}$$

if $x = x_0x_1 \dots x_n \dots$ and $y = y_0y_1 \dots y_n \dots$. Let $w_0 = 1$, $w_1 = w_0 0 w_0$, $\dots$, $w_{n+1} = w_n \underbrace{0 \dots 0}_{n+1 \text{ many}} w_n$. Define the shift map σ by $\sigma(x) = x_1x_2 \dots x_n \dots$. Let x be a sequence which has w_n as its initial sequence for any n. Then x is recurrent. But x is not an almost periodic point.

P, AP, and R are not closed generally (for an example see Fig. 2 in Chapter 4). Let us consider the closure of these sets. We can ask if $\overline{P} = \overline{AP} = \overline{R}$. For continuous map on the interval it has been proved that $\overline{P} = \overline{AP} = \overline{R}$ [Shar, 1964b, Youg, 1979, CovH, 1980, Xio, 1983b]. For the shift map on $\Sigma_p = \{0,\ 1,\ \dots,\ p-1\}^{\ N}$,

$\overline{P} = \overline{AP} = \overline{R} = \Sigma_p$. Generally this is not true. For example, an irrational rotation on the circle S^1 has no periodic point. Then $\overline{P} = \emptyset$, but $\overline{AP} = \overline{R} = S^1$.

A point $x \in X$ is called an ω-limit point of a point $y \in X$ if x is a limit point of $Orb(y)$. The collection of all ω-limit points of y is denoted by $\omega(y)$. The collection of all ω-limit points in X is denoted by $\Lambda(f)$ or Λ. Then $P \subset AP \subset R \subset \Lambda$. For a continuous map f on the interval, Sharkovskii [Shar, 1982] has shown that Λ is closed. Thus $\overline{P} \subset \Lambda$. Let $\Lambda^2 = \Lambda(f|_\Lambda)$, ..., $\Lambda^n = \Lambda(f|_{\Lambda^{n-1}})$, ... and $\Lambda^\infty = \bigcap_{n=1}^\infty \Lambda^n$. Then Λ^∞ is called the attracting center. Xiong [Xio, 1988] has proved that $\Lambda^2 = \cdots = \Lambda^n = \cdots = \Lambda^\infty$ for any continuous map on the interval and $\Lambda^2 \subset \overline{P}$.

A point is called an $\alpha-$limit point of a point y if there exist a sequence $\{y_i\}$ of points in X and a sequence $\{n_i\}$ of positive integers such that $f^{n_i}(y_i) = y$ and $y_i \to x$. The collection of α-limit points of y is denoted by $\alpha(y)$. If x is both an α-limit point and an ω-limit point of y, then x is called an γ-limit point. The collection of all γ-limit points is denoted by Γ(f) or Γ. Xiong [Xio, 1988] has proved that $\Gamma = \Lambda^2$ for continuous maps on the interval. Therefore Γ is the attracting center of f.

A point x is called a nonwandering point if for any neighborhood V of x there is an integer $n > 0$ such that $f^n(V) \cap V \neq \emptyset$. The collection of nonwandering points is denoted by $\Omega(f)$ or Ω. Ω is a closed set. Let $\Omega^2 = \Omega(f|_\Omega)$, ..., $\Omega^n = \Omega(f|_\Omega^{n-1})$ and $\Omega^\infty = \bigcap_{n=1}^\infty \Omega^n$. Then Ω^∞ is called the Birkhoff center of f. For continuous maps on the interval $\Omega^\infty = \Omega^2 = \overline{P}$ [Shar, 1964b, Nit, 1980], and $\Lambda(f) = \bigcap_{n=0}^\infty f^n(\Omega(f))$ [Blok, 1982].

A point x is called a chain recurrent point if for any number $\epsilon > 0$ there exists a finite sequence of points, $x_0 = x, x_1, \ldots, x_{n-1}, x_n = x$, such that $d(f(x_i), x_{i+1}) < \epsilon$ for $i = 0, 1, \ldots, n-1$. The collection of chain recurrent points is denoted by $CR(f)$ or CR. CR is a closed set and $CR(f|CR(f)) = CR(f)$.

We have an inclusion relation as follows.

$$P \subset AP \subset R \subset \Lambda^\infty \subset \cdots \subset \Lambda^n \subset \cdots \subset \Lambda \subset \Omega \subset CR.$$

For continuous maps on the interval, we have

$$P \subset AP \subset R \subset \Gamma = \Lambda^2 = \Lambda^n = \Lambda^\infty \subset \overline{P} \subset \Lambda \subset \Omega \subset CR. \tag{1.1}$$

For continuous maps on the interval, the following are equivalent.

(1) $CR = P$; (2) $\Omega = P$; (3) $\Lambda = P$; (4) $\overline{P} = P$; (5) $R = P$; (6) $AP = P$; (7) for any $x \in I$, $\omega(x)$ is a periodic orbit.

The equivalence of (4) and (7) was proved by Sharkovskii [Shar, 1965], while the equivalence of (2) and (4) was proved by Zhou [Zhou, 1981], Nitecki [Nit, 1982a] and Xiong [Xio, 1981]. The equivalence of (1) and (4) was obtained by Block and Franke [BloF, 1983], and the equivalence of (4) and (5) was anounced by Blokh [Blok, 1982] and was proved by Xiong [Xio, 1985].

For any continuous map on the interval, Coven and Nitecki [CovN, 1981] have shown that $x \in \Omega$ if and only if $x \in \alpha(x)$.

In section 3.1 we will prove the following theorem.

THEOREM 3.1.8. *Let $f : X \to X$ be continuous. Properties* 1) – 6) *hold.*

1) *Let $x \in X$ and let $\underline{x} \in \varprojlim(X, f)$ satisfy $\underline{x} = (x, x_1, x_2, \dots)$. Then $\omega(\underline{x}, \sigma) = \varprojlim(\omega(x, f), f)$.*

2) *$R(\sigma) = \varprojlim(R(f), f)$.*

3) *$AP(\sigma) = \varprojlim(AP(f), f)$.*

4) *$CR(\sigma) = \varprojlim(CR(f), f)$.*

5) *When f is onto X, $\Omega(\sigma) = \varprojlim(\Omega(f), f)$.*

6) *If f is onto X and $\Lambda(\sigma_f)$ is closed, then $\Lambda(\sigma) = \varprojlim(\Lambda(f), f)$.*

Let $f : [0, 1] \to [0, 1]$ be continuous. Then the following property holds.

7) *The condition "f is onto X" in* 5) *and* 6) *is not necessary. Also $\Lambda(\sigma) = \Omega(\sigma)$.*

A counterexample is given to show that 5) and 6) are not true in general if f is not onto X.

Next we consider continuous maps on the interval. Let $T = \{x \in I : f \text{ is not a local homeomorphism at } x\}$ denote the set of turning points of f and $Orb_+(T) = \{y : y = f^i(x), i > 0, x \in T\}$ denote the set of forward images of turning points of f. In chapter 4, we will determine which classes of chain recurrent points (as detemined in the display (1.1)) must be contained in $\overline{Orb_+(T)}$. The main theorems are

THEOREM 4.1.4. $\Omega(f) \setminus \Gamma(f) \subset \overline{Orb_+(T)}$.

THEOREM 4.2.3. $\Omega(f) \setminus \Lambda(f) \subset Orb_+(T)$.

If f has no periodic point with period of the form $m2^n$ with $m > 1$ odd, then we say that f is of Type 2^∞. The collection of Type 2^∞ maps is denoted by $C(2^\infty)$. Xiong [Xio, 1983a] has proved that if $\overline{R(f)} = R(f)$, then $f \in C(2^\infty)$. Chu and Xiong [ChX, 1984] gave an example to show that a Type 2^∞ continuous map f does not satisfy $\overline{R(f)} = R(f)$. The author [Li, 1986] proved that for a Type 2^∞ unimodal map f, $\overline{R(f)} = R(f)$. Later, Xiong [Xio, preprint] proved that a Type 2^∞ piecewise monotone map g satisfies $\overline{R(g)} = R(g)$.

In Chapter 5, we show that $\Lambda(f) = R(f)$ for Type 2^∞ unimodal map f. Note that $\Omega(f)$ may not be equal to $\Lambda(f)$ [Nit, 1982b]. Actually, $\Omega(f) \setminus \Lambda(f)$ can contain an infinite orbit of some interval. We study the structure of Type 2^∞ unimodal maps and some properties of Type 2^∞ continuous maps. The following is the main theorem in Chapter 5.

STRUCTURE THEOREM. *Let f be a Type 2^∞ unimodal map with infinitely many different periods. Let t be the turning point. There is a closed invariant set I^∞ which satisfies the following properties:*

i) For any $x \in I^\infty$, we have $\omega(x) = R \setminus P$; for any $x \in D_\infty = I \setminus I^\infty$, either $\omega(x) = R \setminus P$ or $\omega(x)$ is a periodic orbit.

Also $R \setminus P$ is an infinite minimal set, and it is the unique such set.

ii) I^∞ consists of three classes of components.

1) Some points in $R \setminus P$;

2) Countably many nontrivial closed intervals with endpoints in $\alpha(t) \cap (R \setminus P)$ *and with no interior points in* $Orb_-(t)$.

3) Countably many nontrivial closed intervals, denoted by C_p, $p \geq 0$, *with two endpoints in* $Orb_+(t)$ *and* $\alpha(t) \cap (R \setminus P)$ *respectively, and finitely many sets, denoted by* $C_p, 0 > p \geq M$ *for some integer* $M < 0$, *each of which consists of finitely many closed intervals with endpoints in* $\alpha(t) \cap (R \setminus P)$ *and interior containing a point in* $Orb_-(t)$.

iii) $f|_{I^\infty}$ *is semi-conjugate to the Adding Machine.* $f|_{R \setminus P}$ *is at most two to one semi-conjugate to the Adding Machine.*

iv) There is an integer $M > 0$ *such that* $\Omega = R \cup \{y; y = f^m(t), m \geq M\}$.

CHAPTER 2

ω-CHAOS AND TOPOLOGICAL ENTROPY

In this chapter, we give a new concept of chaos, ω-chaos, and prove some properties of ω-chaos. Then we prove that ω-chaos is equivalent to positive entropy on the interval. We also prove that positive entropy is equivalent to the definition of chaos given by Devaney on the interval.

In Section 2.1, we introduce the definitions of topological entropy, ω-chaos and chaos in the sense of Devaney, and prove a proposition given by Smítál. By this proposition we know that for continuous maps on the interval if we have the first and second conditions in the definition of ω-chaos, then we also have the third condition.

In section 2.2, we prove that the shift maps have ω-chaos. We first show that the one-sided shift map has ω-chaos. Then, as corollaries, we state some results about other shift maps. Suppose $f : X \to X$ and $g : Y \to Y$ are continuous maps. If there is a continuous map $h : X \to Y$ such that $hf(x) = gh(x)$ for all $x \in X$, then we say that f is *semiconjugate to* g and h is a *semiconjugacy*. Moreover, if h maps at most countably many points to one, then we say that f is *at most countable to one semiconjugate to* g and h is a *countable to one semiconjugacy*. In the case of finite to one, we have a similar terminology.

In section 2.3, we prove two general theorems on compact metric spaces. First, if f is countable to one semiconjugate to g, and g satisfies statement II, then f also satisfies statement II. Second, under some conditions, if f^m satisfies statement II for some positive integer m, then f also satisfies statement II. Recall that statement II is given in Theorem 2.4.8 on page 3.

In section 2.4, we prove the equivalence of positive topological entropy and ω-chaos on the interval by using the results of section 2.2 and section 2.3.

In section 2.5, we obtain a general result on a compact metric space. The result demonstrates that if f^m is chaotic in the sense of Devaney for some integer $m > 0$, then f is chaotic in the same sense. Then we prove the equivalence of positive topological entropy and chaos in the sense of Devaney on the interval.

§2.1. Definitions and Preliminaries

In 1965, Adler, Konheim and McAndrew [AKM] gave a definition of topological entropy as follows.

DEFINITION 2.1.1. *Let X be a compact Hausdorff space and let $f : X \to X$ be a continuous mapping. For a cover α of X we denote*

$$\begin{aligned} N(\alpha) &= min\{Card\beta;\ \beta \text{ is a subcover of } \alpha\}, \\ \alpha_n &= \{A_0 \cap f^{-1}(A_1) \cap \cdots \cap f^{-(n-1)}(A_{n-1}); A_0, A_1, \ldots, A_{n-1} \in \alpha\}, \\ h(f,\alpha) &= \lim_{n\to\infty} \frac{1}{n} \log N(\alpha_n). \end{aligned}$$

We define the topological entropy of f by

$$h(f) = sup\{h(f,\alpha); \alpha \text{ is an open cover of } X\}.$$

The following definition is based on the work of Li and Yorke [LiY].

DEFINITION 2.1.2. *Let X be a metric space with metric d, and let $f : X \to X$ be a continuous map. A subset S of X containing no periodic points is called a scrambled set if for any $x, y \in S$ with $x \neq y$,*

1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$;

2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$;

3) $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > 0$ *for any periodic point* p.

We say that f *is chaotic in the sense of Li and Yorke if there is an uncountable scrambled set.*

Let X be a compact metric space. The accumulation points of the orbit $Orb(x, f) = \{y \in X; y = f^m(x)$ for some integer $m > 0\}$ of x under a map $f : X \to X$ are called the ω-limit points, and the set of all these points is denoted by $\omega(x, f)$. Then $\omega(x, f)$ is a closed invariant subset of X.

A subset M of X is called a *minimal set* under f on X if M is a nonempty, closed, invariant set which has no proper nonempty, closed, invariant subset. If there is an n such that M is a minimal set under f^n, but is not a minimal set under f^i for $i = 1, \ldots, n-1$, then we say that M is a *periodic minimal set under* f *with period* n. When M is a periodic minimal set under f with period n, $f(M), \ldots, f^{n-1}(M)$ are also periodic minimal sets under f with period n. The set $\{M, f(M), \ldots, f^{n-1}(M)\}$ is called the *periodic orbit of the minimal set* M.

DEFINITION 2.1.3. *Let* X *be a compact metric space, and let* $f : X \to X$ *be a continuous map. Let* $S \subset X$. *We say that* S *is an* ω*-scrambled set if, for any* $x, y \in S$ *with* $x \neq y$,

1) $\omega(x, f) \setminus \omega(y, f)$ *is uncountable;*

2) $\omega(x, f) \cap \omega(y, f)$ *is nonempty;*

3) $\omega(x, f)$ *is not contained in the set of periodic points.*

We say that f *is* ω*-chaotic if there is an uncountable* ω*-scrambled set.*

When we say that f satisfies *statement II* we mean $\bigcap_{x\in S} \omega(x, f) \neq \phi$.

For maps on the interval, condition 3) is not necessary, since if $\omega(x, f)$ is contained in the set of periodic points then $\omega(x, f)$ is a periodic orbit. In order to prove the property, we need this following proposition. Two set A and B are called *separated* if $\overline{A} \cap \overline{B} = \phi$.

PROPOSITION 2.1.4 [Kuc]. *Let $f : X \to X$ be a map, where X is a topological space, satisfying the first axiom of countability. Let x be in X and $\omega(x, f)$ be nonempty. Let $f|_{\omega(x,f)}$ be continuous. If A is a proper subset of $\omega(x, f)$ seperated from $\omega(x, f) \setminus A$, and both A and $\omega(x, f)$ are compact, then A is not invariant under f.*

The following proof is given by Smítál.[1]

PROPOSITION 2.1.5. *On a compact interval I, for $x \in I$, $\omega(x) \subset P(f)$ implies that $\omega(x)$ is a periodic orbit.*

PROOF: Set $W = \omega(x, f)$. Assume W is infinite. Then $W = \cup_{n=0}^{\infty} F_n$, where $F_n = \{y \in W : f^n(y) = y\}$. Since W is compact and f is continuous, every F_n is compact. Since W is a complete metric space, by Baire's Theorem, it must be of Baire's Second Category. Thus there is a smallest integer $n \neq 1$ such that F_n is dense in $J \cap W \neq \phi$, where J is an interval. Since F_n is closed, $J \cap W = J \cap F_n$. Moreover since W is nowhere dense in I, there is an open interval $J_1 \subset J$ with $W^* = J_1 \cap W \neq \phi$ such that

$$(1) \qquad d(W^*, W \setminus W^*) = \inf\{d(x, y) : x \in W^*, y \in W \setminus W^*\} > 0.$$

Now set $W_0 = \cup_{i=1}^{n} f^i(W^*) = Orb(W^*)$ and $W_1 = W \setminus W_0$. Then by (1) we get $d(W_0, W_1) > 0$. Thus W_1 is closed and invariant. By Proposition 2.1.4, W_1 has to be empty. So, $W = W_0$ has only finitely many periods of periodic points.

Let $y \in W$ and $V = (y - \epsilon, y + \epsilon)$ be an open interval containing y. Since W is nowhere dense in V, there is an open interval $U \subset V$ such that $d(U \cap W, W|_{U \cap W}) > 0$. Then, by the same argument as above, we have $W = \cup_{i=1}^{n} f^i(U \cap W)$. By letting $\epsilon \to 0$ we obtain $W = orb(y)$. Therefore we have proved the proposition. ∎

It is easy to see that $\omega(x, f)$ in 1) and 2) of Definition 2.1.3 can be replaced by $\overline{Orb(x, f)}$. Furthermore the second condition of Definition 2.1.2 implies the second

[1]We know the proof by personal communication.

condition of Definition 2.1.3, and the third condition of Definition 2.1.3 implies the third condition of Definition 2.1.2.

DEFINITION 2.1.6. *A continuous map $f : X \to X$ is called topologically transitive if for any pair of open sets $U, V \subset X$ there exists $k > 0$ such that $f^k(U) \cap V \neq \phi$.*

DEFINITION 2.1.7. *Let X be a metric space. A continuous map $f : X \to X$ has sensitive dependence on initial conditions on X if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$.*

This definition is equivalent to saying that there is a $\delta > 0$ such that, for any $x \in X$, there exist a sequence $\{y_k\}$ of points in X and a sequence $\{n_k\}$ of positive integers such that $\lim_{k\to\infty} y_k = x$ and $d(f^{n_k}(y_k), f^{n_k}(x)) > \delta$.

DEFINITION 2.1.8. *Let X be a metric space. Let $f : X \to X$ be a continuous map. f is said to be chaotic in the sense of Devaney if there is a nonempty, closed, invariant set $D \subset X$ such that the following conditions are satisfied.*

1. *$f|_D$ is topologically transitive.*
2. *$f|_D$ has sensitive dependence on initial conditions.*
3. *The periodic points of f in D are dense in D.*

The set D is called a chaotic set in the sense of Devaney.

§2.2. Statement II for Shift Maps

Define Σ to be the set of sequences $a_0a_1a_2 \ldots a_n \ldots$, where $a_n = 0$ or 1 with the topology defined by the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}$$

if $x = x_0x_1 \ldots x_n \ldots$ and $y = y_0y_1 \ldots y_n \ldots$. Then Σ is a compact metric space.

Define the 'one-sided shift' σ on Σ by $\sigma(a_0a_1a_2\dots) = a_1a_2\dots$. It is easy to show that σ is continuous.

Let $\mathcal{I}$ denote the set of the irrational numbers on $[0, \frac{1}{2}]$.

Let S^1 be the unit circle. Let $r_\alpha(\alpha \in \mathcal{I})$ denote the rotation by $2\pi\alpha$ radians on S^1.

Let $[a, b]$ denote the closed arc on S^1 from a counterclockwise to b with length $2\pi\alpha$. Then $r_\alpha(a) = b$ by the definition of r_α.

For any $x \in S^1$, define

$$h_\alpha(x) = x_0x_1x_2\dots x_n\dots$$

by the following rule: *if* $r_\alpha^n(x) \notin [a, b]$, *we define* $x_n = 0$; *if* $r_\alpha^n(x) \in (a, b)$, *we define* $x_n = 1$; *if* $r_\alpha^n(x) = a$, *then we define* $x_n = 0$ *or* 1 *but* $x_n \neq x_{n+1}$; *if* $x = b$, *then* $x_0 = 0$ *or* 1. We denote $(h_\alpha(x))_n = x_n$ for all $n \geq 0$.

h_α maps $x \in S^1$ to at most two points, since α is an irrational number, and so the points in the orbit of x under r_α never repeat. $h_\alpha(x)$ has two images iff $r_\alpha^n(x) = a$ for some $n \geq 0$ or $x = b$. If s is one of the two images of $h_\alpha(x)$, we still write $s = h_\alpha(x)$ instead of $s \in h_\alpha(x)$. We say that x is a 1-image (2-images) point if $h_\alpha(x)$ is unique (not unique).

We will prove first that there are uncountably many disjoint minimal sets in Σ. Then, for each minimal set, we construct a point in Σ whose ω-limit set contains the minimal set. Then we will show that these points form an ω-scrambled set.

LEMMA 2.2.1. h_α *is continuous at a 1-image point. If* x *is a 2-images point with* $r_\alpha^n(x) = a$ *point and* $(h_\alpha(x))_n = 0$ *(or* 1*) for some* $n \geq 0$, *then* h_α *is left (right) continuous at* x. *If* $x = b$ *and* $h_\alpha(x) = 0$ *(or* 1*), then* h_α *is right (left) continuous at* x.

PROOF: For any $\epsilon > 0$, there exist N such that

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \epsilon.$$

Suppose $x \in S^1$ such that $r_\alpha^n(x) \neq a$ or b for all $n \geq 0$. Then $\{r_\alpha^n(x) : n = 0, 1, 2, \dots\}$ is contained in $(a, b) \cup (b, a)$ which is an open set.

Let x^m approach to x on S^1. Because r_α^n are continuous for $n = 0, 1, 2, \dots$, for given $N > 0$, there is an L such that, as $m > L$, $r_\alpha^n(x)$ and $r_\alpha^n(x^m)$ are in (a, b) or (b, a) simultaneously for $n = 0, 1, 2, \dots, N$. Then

$$(h_\alpha(x^m))_n = (h_\alpha(x))_n$$

for $n = 0, 1, 2, \dots, N$. Therefore,

$$\begin{aligned} d(h_\alpha(x^m)_n, h_\alpha(x)) &= \sum_{n=0}^{\infty} \frac{|x_n^m - x_n|}{2^n} \\ &= \sum_{n=N+1}^{\infty} \frac{|x_n^m - x_n|}{2^n} \\ &\leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} \\ &< \epsilon. \end{aligned}$$

So, h is continuous at x which is a 1-image point. Suppose $x \in S^1$ is a 2-images point with $(h_\alpha(x))_{n_0} = 0$, where $n_0 = 0$ satisfies $r_\alpha^{n_0}(x) = a$. Then there is a sequence x^m with $x^m \in (b, a)$ and $x^m < x$ such that $\lim_{m\to\infty} x^m = x$. Since r_α^n are uniformly continuous for $n = 0, 1, 2, \dots, n_0 - 1$, $r_\alpha^n(x^m)$ and $r_\alpha^n(x)$ simultaneously belong to (a, b) or (b, a) for $n = 0, 1, 2, \dots, n_0 - 1$. Then $(h_\alpha(x^m))_n = h_\alpha(x))_n$ for $n = 0, 1, 2, \dots, n_0 - 1$.

Since the arc length between x^m and x is the same as the arc length between $r_\alpha^n(x^m)$ and $r_\alpha^n(x)$, there is an $L_2 > L_1$ such that, as $m > L_2$, the arc length between

$r_\alpha^n(x^m)$ and $r_\alpha^n(x)$ is less than the arc length between b and a for $m = 0, 1, 2, \ldots, n_0$. Since r_α^n preserves order, we can see that $r_\alpha^{n_0}(x^m) \in (b, a)$. That is to say that $(h_\alpha(x^m))_{n_0} = 0$. As $m > L_2$,

$$d(h_\alpha(x^m), h_\alpha(x)) = \sum_{n=n_0+2}^{\infty} \frac{|(h_\alpha(x^m))_n - (h_\alpha(x))_n|}{2^n}.$$

Note that $r_\alpha^{n_0+2}(x)$ is a 1-image point. Given any $\epsilon > 0$,there is an $L_3 > L_2$ such that, if $m > L_3$, we have

$$d(h_\alpha(r_\alpha^{n_0+2}(x^m)), h_\alpha(r_\alpha^{n_0+2}(x))) < \epsilon.$$

Then, if $m > L_3$,

$$\begin{aligned} d(h_\alpha(x^m), h_\alpha(x)) &= \sum_{n=n_0+2}^{\infty} \frac{|(h_\alpha(x^m))_n - (h_\alpha(x))_n|}{2^n} \\ &= \sum_{n=0}^{\infty} \frac{|(h_\alpha(r_\alpha^{n_0+2}(x^m)))_n - (h_\alpha(r_\alpha^{n_0+2}(x)))_n|}{2^n} \\ &< \epsilon. \end{aligned}$$

By using similar discussion, we can complete the rest of the proof. ∎

The following lemma is well known.

LEMMA 2.2.2. *For any irrational number $\alpha > 0$, the set $\{t \in [0,1]; t = n\alpha (mod\ \mathbf{N})$, for some $n \in \mathbf{N}\}$ is dense on $[0,1]$, where $\mathbf{N}$ is the set of the natural numbers. Also, for any $x \in S^1$, $\{r_\alpha^n(x); n \geq 0\}$ (and $\{r_\alpha^n(x); n < 0\}$) is dense on S^1.*

The following lemma follows from the definition of h_α.

LEMMA 2.2.3. *$h_\alpha r_\alpha(x) = \sigma h_\alpha(x)$ for all $x \in S^1$ in the sense that $h_\alpha(x)$ is a set of at most two points.*

Let us denote $h_\alpha(S^1)$ by M_α. Then we have the following proposition.

PROPOSITION 2.2.4. *M_α is a minimal set in Σ under σ.*

PROOF: *1) M_α is closed.* Take a sequence $s_n \in M_\alpha \subset \Sigma$ such that $\lim_{n\to\infty} s_n = s$ with $s \in \Sigma$. We will prove that $s \in M_\alpha$.

By the definition of M_α, there exists a sequence x^n on S^1 so that $h_\alpha(x^n) = s_n$. The sequence x^n has an accumulation point on S^1. Without loss of generality, suppose $\lim_{n\to\infty} x^n = x$ (if x is a 2-images point, take x^n to be a monotone sequence in a neighborhood of x). By Lemma 2.2.1, $\lim_{n\to\infty} h_\alpha(x_n) = h_\alpha(x)$, i.e $\lim_{n\to\infty} s_n = h_\alpha(x)$. So $s = h_\alpha(x) \in M_\alpha$.

2) M_α is invariant. If $s \in M_\alpha$ and $s = h_\alpha(x)$ with $x \in S^1$. Then, by Lemma 2.2.3, $\sigma(h_\alpha(x)) = h_\alpha(r_\alpha(x)) \in M_\alpha$, i.e $\sigma(s) \in M_\alpha$. So M_α is invariant.

3) M_α is minimal. Take any $s \in M_\alpha$. We are going to show that $\overline{orb(s,\sigma)} = M_\alpha$ which implies that M_α is a minimal set. By the definition of M_α, there is an $x \in S^1$ such that $h_\alpha(x) = s$.

We claim that $\{\sigma_\alpha^n(s) : n \geq 0\}$ is dense on M_α. In fact, for any $s^* \in M_\alpha$, there is a $y \in S^1$ such that $h_\alpha(y) = s^*$. By Lemma 2.2.2, $\{r_\alpha^n(x);\ n \geq 0\}$ is dense on S^1. Then there is a sequence $y_m \in \{r_\alpha^n(x);\ n \geq 0\}$ such that $\lim_{m\to\infty} y_m = y$ with y_m monotone. By Lemma 2.2.1, $\lim_{m\to\infty} h_\alpha(y_m) = h_\alpha(y) = s^*$. By Lemma 2.2.3, we know that $h_\alpha(r_\alpha^n(x)) = \sigma^n(h_\alpha(x)) = \sigma^n(s)$. Thus $h_\alpha(y_m) \in \{\sigma_\alpha^n(s) : n \geq 0\}$. Therefore, $s^* \in \overline{orb(s,\sigma)}$. Then we have proved that $M_\alpha \subset \overline{orb(s,\sigma)}$. But $M_\alpha \supset \overline{orb(s,\sigma)}$ since M_α is closed invariant. So $\overline{orb(s,\sigma)} = M_\alpha$. ∎

One can find another proof of this proposition in Hedlund [Hed]. The construction here is the same as the construction in Hedlund [Hed] except that we use circle instead of a real line. In Hedlund [Hed], this kind of minimal set is called a *Sturmian minimal set.*

PROPOSITION 2.2.5. *M_α is uncountable.*

PROOF: If we can prove that $h_\alpha(x) \neq h_\alpha(y)$ for $x \neq y$ in $(a,b) \setminus \{r_\alpha^{-n}(b) : n \geq 0\}$, then M_α is uncountable since (a,b) contains uncountably many points and $\{r_\alpha^{-n}(b) : n \geq 0\}$ is countable.

By Lemma 2.2.2, $\{r_\alpha^{-n}(b) : n \geq 0\}$ is dense on S^1. Therefore, there is an n_0 such that $x < r_\alpha^{-n_0}(a) < y$, i.e $r_\alpha^{n_0}(x) < a < r_\alpha^{n_0}(y)$. Also we have $r_\alpha^{n_0}(y) < b$ since the arc length of $(r_\alpha^{n_0}(x), r_\alpha^{n_0}(y))$ equals the arc length of (x,y) which is less than $2\pi\alpha$—the arc length of (a,b). Then we can see that

$$(h_\alpha(x))_{n_0} \neq (h_\alpha(y))_{n_0}.$$

Thus, $h(x) \neq h(y)$. ∎

PROPOSITION 2.2.6. *For $\alpha \neq \beta$ in $\mathcal{I}$, $M_\alpha \cap M_\beta = \phi$. Thus there is an uncountable collection of pairwise disjoint minimal sets of Σ under the one-sided shift map σ.*

PROOF: Let $x, y \in S^1$. Let $T_\alpha(x,n) = [n\alpha]$ which is the greatest integer less than or equal to $n\alpha$. Denote $I_\alpha(x,n)$ to be the number of ones among $(h_\alpha(x))_0, \ldots, (h_\alpha(x))_n$. By the definition of h_α, it is easy to see that

$$|T_\alpha(x,n) - I_\alpha(x,n)| \leq 2.$$

Then

$$\lim_{n\to\infty} \frac{I_\alpha(x,n)}{n} = \lim_{n\to\infty} \frac{T_\alpha(x,n)}{n} = \alpha.$$

Thus, for any pair of irrational numbers $\alpha \neq \beta$, there exists a positive integer n big enough that

$$I_\alpha(x,n) \neq I_\beta(y,n).$$

That implies that $h_\alpha(x) \neq h_\beta(y)$ for any $x, y \in S^1$ and irrational numbers $\alpha \neq \beta$. Then $M_\alpha \cap M_\beta = \phi$.

Because $\mathcal{I}$ is uncountable, $\{M_\alpha : \alpha \in \mathcal{I}\}$ is uncountable. Therefore, there are uncountably many disjoint minimal sets. ∎

REMARK. *The second sentence of Proposition 2.2.6 was also proved in [MorH].*

For any $s \in M_\alpha$, $s = s_0 s_1 s_2 \ldots$, define $t = t(s) \in \Sigma$ by

$$t = 0 s_0 0 0 s_0 s_1 0 0 0 s_0 s_1 s_2 0 \ldots \overbrace{0 \ldots 0}^{n \text{ many}} s_0 s_1 \ldots s_{n-1} 0 \ldots .$$

Denote t by $\langle 0, s \rangle$, and let $\underline{0} = 00 \ldots 0 \ldots$. Set

$$\mathcal{L} = \mathcal{L}(s) = \{x \in \Sigma : \ x = s_i \ldots s_n 0 \ldots 0 \ldots \text{ where } \ n \geq i \geq 0\},$$
$$\mathcal{O} = \mathcal{O}(s) = \{x \in \Sigma : \ x = \overbrace{0 \ldots 0}^{k \text{ many}} s_0 s_1 s_2 \ldots \text{ where } \ k \geq 1 \text{ and } \ n \geq 1\}.$$

If we take a different s in M_α, we may get a different $\mathcal{L}$ and $\mathcal{O}$. But the following properties do not depend on the choice of s. The reason is that M_α is a minimal set. This remark will be clear after we finish this section.

LEMMA 2.2.7. $\omega(t, \sigma) = \{\underline{0}\} \cup M_\alpha \cup \mathcal{L} \cup \mathcal{O}$.

PROOF: It is obvious that

$$\{\underline{0}\} \cup M_\alpha \subset \omega(t, \sigma)$$

since $\underline{0}$ and s are in $\omega(t, \sigma)$. Because $s \in \omega(s, \sigma)$, for any $u = s_i \ldots s_n \cdots \in \mathcal{L}$, there is a sequence $\{n_j\}$ of positive integers such that

$$d(\sigma^{n_j}(s), u) < \frac{1}{2^{n-i+1}},$$

i.e $(\sigma^{n_j}(s))_p = u_p$ for $1 \leq p \leq n - i + 1$. It follows that there is a sequence $\{m_j\}$ of positive integers such that $(\sigma^{m_j}(t))_p = u_p$ for $1 \leq p \leq n - i + 1$ and $(\sigma^{m_j}(t))_q = 0$ for $n - i + 2 \leq q \leq M(j)$ where the positive integer $M(j)$ goes to infinity as j goes to infinity. Therefore $\lim_{j \to \infty} \sigma^{m_j}(t) = u$. Hence $\mathcal{L} \subset \omega(t, \sigma)$. Similarly, $\mathcal{O} \subset \omega(t, \sigma)$ since $s \in \omega(s, \sigma)$.

Next we prove $\omega(t,\sigma) \subset \{\underline{0}\} \cup M_\alpha \cup \mathcal{L} \cup \mathcal{O}$.

Given $x = x_0x_1 \dots x_n \dots$, we call $x_i \dots x_j$ a block of x with length $j - i + 1$ for $j \geq i \geq 0$.

Note that $x \in \omega(t,\sigma)$ iff for any n each block of x with length n matches infinitely many blocks of t.

In the following discussion, when we say "a block of s in t" we mean $s_0s_1 \dots s_n$ for some $n \geq 0$ between two blocks of zeros. When we say "a block of $\underline{0}$ in t" we mean $\overbrace{0 \dots 0}^{n \text{ many}}$ between two blocks of s in t.

Let $x \in \omega(t,\sigma)$. We will prove that $x \in \{\underline{0}\} \cup M_\alpha \cup \mathcal{L} \cup \mathcal{O}$.

If $x \in \omega(s,\sigma) \cup \{\underline{0}\} = M_\alpha \cup \{\underline{0}\}$, it is trivial. If not, we can suppose that the initial block of x with length $N+1$ is not a block of s. We can take N to be the smallest integer such that $x_0x_1 \dots x_N$ is not a block of s. Since x_0 is always a block of s, then $N > 0$. Also $x_0x_1 \dots x_{N-1}$ is a block of s.

For any $n \geq N$, $x_0x_1 \dots x_N \dots x_n$ matches infinitely many blocks of t. Then there is a sequence of positive integers $j_i(n) > n$ so that $x_0x_1 \dots x_N \dots x_n$ is a block of $\overbrace{0 \dots 0}^{j_i(n) \text{ many}} \; s_0s_1 \dots s_{j_i(n)-1} \; \overbrace{0 \dots 0}^{j_i(n)+1 \text{ many}}$.

Since $x_0x_1 \dots x_N$ is neither a block of $\underline{0}$ nor a block of s, there are only two possibilities.

1) $x_0x_1 \dots x_{k_i} = s_{j_i(n)-k_i-1} \dots s_{j_i(n)-1}$ is a block of length $k_i + 1$ consisting of the last k_i+1 digits of $s_0s_1 \dots s_{j_i(n)-1}$, and $x_j = 0$ for $j = k_i+1, \dots, n$ with $k_i < N$.

2) $x_0x_1 \dots x_{k_i-1} = \overbrace{0 \dots 0}^{k_i \text{ many}}$ and $x_{k_i} \dots x_N \dots x_n = s_0s_1 \dots s_{n-k_i}$.

First, suppose 1) happens for infinitely many n. Then there must be an integer k such that $k_i = k$ for infinitely many k_i. Hence x can be written as $s_i \dots s_{i+k} \dots$. Thus $x \in \mathcal{L}$. Finally, if 2) happens for infinitely many n, then by a similar argument we know that $x \in \mathcal{O}$.

Therefore,

$$\omega(t, \sigma) \subset \{\underline{0}\} \cup M_\alpha \cup \mathcal{L} \cup \mathcal{O}.$$

∎

LEMMA 2.2.8. *$\omega(t, \sigma)$ contains only two minimal sets.*

PROOF: For any $x \in \mathcal{L} \cup \mathcal{O}$, there is an N such that $\sigma^n(x) \in \{\underline{0}\} \cup M_\alpha$ for all $n \geq N$. That implies that any closed invariant set which contains x must contain $\{\underline{0}\}$ or M_α. Therefore there is no other minimal set beside $\{\underline{0}\}$ and M_α. ∎

THEOREM 2.2.9. *σ satisfies statement II.*

PROOF: Define $S = \{t_\alpha : \ t_\alpha =< 0, s_\alpha > \text{ for some } s_\alpha \in M_\alpha, \alpha \in \mathcal{I}\}$.

Obviously, S is uncountable by Proposition 2.2.6. We are going to check that S satisfies the three conditions in the definition of ω-chaos and $\cap_{\alpha \in \mathcal{I}} \omega(t_\alpha, \sigma) \neq \phi$.

1) For $\alpha \neq \beta$, we claim that

$$M_\alpha \cap \omega(t_\beta, \sigma) = \phi.$$

If not, then $M_\alpha \subset \omega(t_\beta, \sigma)$. Then by Lemma 2.2.8, $M_\alpha = M_\beta$. But by Proposition 2.2.6, $M_\alpha \cap M_\beta = \phi$. Thus the claim is true. By Lemma 2.2.7 and the claim, we have

$$\omega(t_\alpha, \sigma) \setminus \omega(t_\beta, \sigma) \supset M_\alpha.$$

So, $\omega(t_\alpha, \sigma) \setminus \omega(t_\beta, \sigma)$ is uncountable by Proposition 2.2.5.

2) $\cap_{\alpha \in \mathcal{I}} \omega(t_\alpha, \sigma) \supset \{\underline{0}\}$ is not empty.

3) Since M_α is an uncountable minimal set, $\omega(t_\alpha, \sigma)$ is not contained in the set of periodic points.

Therefore, σ has an ω-scrambled set on Σ which satisfies $\cap_{\alpha \in \mathcal{I}} \omega(t_\alpha, \sigma) \neq \phi$. So, the statement II is satisfied. ∎

Define Σ_p to be the collection of all sequences $x = x_0x_1x_2 \dots x_n \dots$ ($x_n \in \{0, 1, \dots, p\}$) with topology induced by the metric

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}$$

where $x = x_0x_1x_2 \dots$ and $y = y_0y_1y_2 \dots$ are in Σ_p.

Since Σ is a subspace of Σ_p, we have the following property.

COROLLARY 2.2.10. *One-sided shift maps on Σ_p satisfy statement II.*

Instead of working with one-sided shifts, one can work with two-sided shifts to get a similar result.

COROLLARY 2.2.11. *Two-sided shift maps satisfy statement II.*

§2.3. Some Properties of ω-Chaos

In this section, we are going to prove some properties of ω-chaos which will be used in the next section.

Let X be a compact metric space. Let Y be a metric space. Let $g : X \to X$ and $p : Y \to Y$ be continuous maps.

LEMMA 2.3.1. *If g and p are semiconjugate, i.e, if there is a continuous onto map $h : X \to Y$ such that $hg = ph$, then $h(\omega(x, g)) = \omega(h(x), p)$ for any $x \in X$.*

PROOF: For any $y \in \omega(x, g)$, there exists a sequence $\{n_i\}$ of positive integers such that $\lim_{i\to\infty} g^{n_i}(x) = y$. Since h is continuous,

$$\lim_{i\to\infty} p^{n_i}(h(x)) = \lim_{i\to\infty} h(g^{n_i}(x)) = h(y),$$

i.e, $h(y) \in \omega(h(x), p)$. So, $h(\omega(x, g)) \subset \omega(h(x), p)$.

For any $z \in \omega(h(x), p)$, there is a sequence $\{m_i\}$ of positive integers such that

$$\lim_{i \to \infty} p^{m_i}(h(x)) = z.$$

By the property of semiconjugacy, $\lim_{i \to \infty} h(g^{m_i}(x)) = z$. Since X is a compact metric space, $\{g^{m_i}(x)\}$ has a limit point $y \in \omega(x, g)$. By the continuity of h, we have $h(y) = z$. This shows that $\omega(h(x), p) \subset h(\omega(x, g))$. ∎

THEOREM 2.3.2. *If g is countable to one semiconjugate to p with semiconjugacy $h : X \to Y$, then p satisfies statement II implies that g satisfies statement II. Also we can take an ω-scrambled set in X from the preimage under h of some ω-scrambled set in Y.*

PROOF: Since p satisfies statement II, there is an uncountable ω-scrambled set $S(p)$ in Y with $\bigcap_{y \in S(p)} \omega(y, p) \neq \phi$. Let $y_0 \in \omega(y, p)$ for all $y \in S(p)$. Let $S(g)^* = \{x \in X; h(x) \in S(p)\}$. Then we claim that $h^{-1}(y_0) \bigcap \omega(x, g) \neq \phi$ for every $x \in S(g)^*$.

As a matter of fact,

$$y_0 \in \omega(h(x), p) = h(\omega(x, g))$$

for every $x \in S(g)^*$ by Lemma 2.3.1. Thus, for any $x \in S(g)^*$, there is an $x^* \in \omega(x, g)$ such that $h(x^*) = y_0$. That is $x^* \in \omega(x, g) \bigcap h^{-1}(y_0)$. This proves the claim.

Since $h^{-1}(y_0)$ is countable, there must be one point $x_0 \in h^{-1}(y_0)$ such that $x_0 \in \omega(x, g)$ for uncountably many $x \in S(g)^*$. Let $\widetilde{S(g)} = \{x \in S(g)^*; x_0 \in \omega(x, g)\}$. Then $\widetilde{S(g)}$ is uncountable. Define a relation '$\sim$' as follows:

$$x \sim x' \text{ iff } h(x) = h(x').$$

Obviously, '$\sim$' is an equivalent relation. Let $S(g)$ consist of one representative of each equivalence class. Then $S(g)$ is uncountable since $h^{-1}(y)$ is countable for every

$y \in S(g)$. By the definition of $S(g)$,

$$\bigcap_{x \in S} \omega(x, f) \neq \phi.$$

Next, we prove that $S(g)$ satisfies the first condition of the ω-scrambled set. We claim that

$$h(\omega(x,g) \setminus \omega(\widetilde{x},g)) \supset \omega(h(x),p) \setminus \omega(h(\widetilde{x}),p)$$

for every pair $x \neq \widetilde{x}$ in $S(g)$.

For any $y \in \omega(h(x),p) \setminus \omega(h(\widetilde{x}),p)$, there is a $z \in \omega(x,g)$ such that $h(z) = y$ by Lemma 2.3.1. If $z \in \omega(\widetilde{x},g)$, then $h(z) \in h(\omega(\widetilde{x},g)) = \omega(h(\widetilde{x}),p)$. So, $z \in \omega(x,g) \setminus \omega(\widetilde{x},g)$. Thus we have proved the claim.

Since $h(x), h(\widetilde{x}) \in S(f)$ are different, $\omega(h(x),p) \setminus \omega(h(\widetilde{x}),p)$ is uncountable. Then $h(\omega(x,g) \setminus \omega(\widetilde{x},g))$ is uncountable, and so $\omega(x,g) \setminus \omega(\widetilde{x},g)$ is also uncountable. This proves that $S(g)$ satisfies the first condition of ω-scrambled set.

By Lemma 2.3.1, $\omega(x,g)$ is contained in the set of periodic points implies that $\omega(h(x),p)$ is contained in the set of periodic points. So $S(g)$ satisfies the third condition of ω-scrambled set.

Thus we have proved that g satisfies statement II. ∎

LEMMA 2.3.3 [Cop, 1984, Xig, 1986]. *Let X be a compact metric space. If $f : X \to X$ is continuous, then*

$$\omega(x,f) = \bigcup_{i=0}^{n-1} f^i(\omega(x,f^n)) = \bigcup_{i=0}^{n-1} \omega(f^i(x),f^n)$$

and

$$f^i(\omega(x,f^n)) = \omega(f^i(x),f^n)$$

for any positive integer n and $i = 0,1,2,...,n-1$.

Let X be a compact metric space. Let $X_0, X_1, ..., X_{n-1}$ be compact subspaces of X which are invariant under f^n. Suppose $f : X \to X$ is a continuous map such

that $f(X_i) \subset X_{i+1(mod(n))}$ for each $i = 0, \ldots, n-1$. Then we have the following properties.

LEMMA 2.3.4. *Let $\mathcal{M}_i$ denote the collection of minimal sets of X_i under f^n. Then f^{j-i} sets up a one to one correspondance between $\mathcal{M}_i$ and $\mathcal{M}_j$ for given positive integers i and j with $i < j$. Also, the corresponding minimal sets $M_i \in \mathcal{M}_i$ and $M_j \in \mathcal{M}_j$ are in the same periodic orbit of a minimal set.*

PROOF: For any $M_j \in \mathcal{M}_j$, $f^{n-j+i}(M_j)$ is a minimal set under f^n and

$$f^{n-j+i}(M_j) \subset X_i.$$

So $f^{n-j+i}(M_j) \in \mathcal{M}_i$. Let $M_i = f^{n-j+i}(M_j)$. Then

$$f^{j-i}(M_i) = f^{j-i}(f^{n-j+i})(M_j) = f^n(M_j) = M_j.$$

The last step is true because M_j is a minimal set. Thus f^{j-i} maps $\mathcal{M}_i$ onto $\mathcal{M}_j$.

If there is an $M_i' \in \mathcal{M}_i$ such that $f^{j-i}(M_i') = M_j$, then

$$f^{n-j+i}(f^{j-i}(M_i')) = f^n(M_i') = M_i'$$

and

$$f^{n-j+i}(M_j) = M_i \text{ implies that } M_i = M_i'.$$

So f^{j-i} is also one to one. Then f^{j-i} acts on $\mathcal{M}_i$ as a one to one correspondence.

The last part of the lemma follows immediately. ∎

THEOREM 2.3.5. *Suppose that f^m satisfies statement II, and let $S(f^m)$ be an ω-scrambled set as in statement II. Suppose also that for any $x \in S(f^m)$ the following conditions are satisfied.*

1) $\omega(x, f^m)$ contains a finite, nonzero number of infinite minimal sets.

2) $\omega(x, f^m)$ contains only countably many points which are not in these minimal sets.

Then f satisfies statement II.

PROOF: For any $x \in S(f^m)$ and any f^m-minimal set M either $M \subset \omega(x, f)$ or $M \cap \omega(x, f) = \phi$. By the above two lemmas and hypothesis 1), $\omega(x, f)$ contains only finitely many f^m-minimal sets. Since $S(f^m)$ is uncountable, there exists an uncountable subset $S_1(f^m)$ such that $\omega(x, f)$ contains the same number of f^m-minimal sets for each $x \in S_1(f^m)$.

For $x, y \in S_1(f^m)$, say $x \sim y$ if $\omega(x, f)$ and $\omega(y, f)$ contain the same f^m-minimal sets. It is easy to see that this is an equivalence relation. Note that for distinct x and y, $\omega(x, f^m) \setminus \omega(y, f^m)$ contains an infinite f^m-minimal set by Definition 2.1.2 and hypothesis 2). Thus each equivalence class is finite. Let $S(f)$ be a subset of $S_1(f^m)$ which contains exactly one representative of each equivalence class. Then $S(f)$ is uncountable. Also for any pair of distinct points $x, y \in S(f)$, $\omega(x, f) \setminus \omega(y, f)$ contains an infinite minimal set and hence is uncountable. We leave the rest of the proof to the reader. ∎

COROLLARY 2.3.6. *If there is a compact subset D of X and an integer $m > 0$ such that f^m is invariant on D and finite to one semiconjugate to the one-sided shift σ on $\Sigma_p (p > 0)$, then f satisfies statement II.*

PROOF: By the Corollary 2.2.10, σ satisfies statement II, and, for every s in the ω-scrambled set $S(\sigma)$, $\omega(s, \sigma)$ contains only one infinite minimal set. Since f^m is finite to one semiconjugate to σ, f^m satisfies statement II and the ω-scrambled set $S(f^m)$ is taken from the preimage of $S(\sigma)$ under the semiconjugacy by the Theorem 2.3.2. Because two different minimal sets are disjoint and the semiconjugacy maps finite points to one, for every $x \in S(f^m)$, $\omega(x, f^m)$ contains only finitely many infinite minimal sets. ∎

§2.4. Positive Entropy and ω-Chaos on the Interval

In this section, we will work on the interval $I = [0, 1]$. We suppose that $f : I \to I$ is a continuous map. Let $C(2^\infty)$ denote the class of continuous maps with no periodic points of periods not a power of two. We say that x is an almost periodic point of f if for any open set U containing x there is a sequence $\{n_i\}$ of positive integers with bounded gap such that $f^{n_i}(x) \in U$. Then by a classical theorem [Bir, p.199, Got], x is the almost periodic point of f if and only if $\overline{Orb(x)}$ is minimal. Apparently $\overline{Orb(x)}$ is minimal if and only if x is a recurrent point and $\omega(x, f)$ is minimal. Let P, AP, R and Λ denote the sets of periodic points, almost periodic points, recurrent points and ω-limit points respectively. Let $\Lambda^2 = \cup_{x\in\Lambda}\omega(x, f)$. Then we know the following properties.

PROPOSITION 2.4.1 [Xio, 1986, Shar, 1982]. *$\omega(x, f)$ contains only one minimal set for $f \in C(2^\infty)$.*

PROPOSITION 2.4.2 [Xio, 1988]. *$\Lambda \setminus \Lambda^2$ is countable for any continuous map f.*

PROPOSITION 2.4.3 [Xio,1988]. *$\Lambda^2 = R = AP$ for $f \in C(2^\infty)$.*

LEMMA 2.4.4. *For $f \in C(2^\infty)$ and $x \in I$, if $\omega(x) \subset \Lambda^2$, then $\omega(x)$ is a minimal set.*

PROOF: For any $y \in \omega(x)$, we have $\omega(y) \subset \omega(x)$ since $\omega(x)$ is invariant. Then $y \in \omega(x) \subset \Lambda^2 = AP$ by Proposition 2.4.3. Thus $\omega(y)$ is a minimal set.

Fix $z_0 \in \omega(x)$. Then $\omega(z_0)$ is a minimal set which contains z_0 and $\omega(z_0) \subset \omega(x)$. By Proposition 2.4.1, $\omega(z_0) = \omega(y)$ for every $y \in \omega(x)$. So $y \in \omega(y) = \omega(z_o)$ for every $y \in \omega(x)$. Thus $\omega(x) = \omega(z_0)$ and $\omega(x)$ is a minimal set. ∎

Block has proved [Blo, 1978a] that if f has zero entropy then f is a map in $C(2^\infty)$, and Misiurewicz [Mis] has proved the converse, that is, if f has positive entropy, then f is not in $C(2^\infty)$. Later, Xiong [Xio, 1989] gaved an easier proof of Misiurewicz's result. Combining their results, we have the following Proposition.

PROPOSITION 2.4.5. *$f \in C(2^\infty)$ iff f has zero entropy.*

By the above properties, we can prove the following property.

PROPOSITION 2.4.6. *If there are two points y, z satisfying the conditions of $\omega-$ scrambled set, then f has positive entropy.*

PROOF: Suppose that f does not have positive entropy, then, by Proposition 2.4.5, $f \in C(2^\infty)$.

If one of $\omega(y)$ and $\omega(z)$ is contained in Λ^2, then, by Lemma 2.4.4, if $\omega(y)$ and $\omega(z)$ have nonempty intersection, then one is contained in the other. Hence, the first two conditions of Definition 2.1.3 cannot both hold. So, both $\omega(y) \cap (\Lambda \setminus \Lambda^2)$ and $\omega(z) \cap (\Lambda \setminus \Lambda^2)$ are nonempty.

By Proposition 2.4.1, $\omega(y)$ contains only one minimal set $M(y)$. Since we can find some $x \in M(y)$ such that $\omega(x) = M(y)$, $M(y)$ is contained in Λ^2. We claim that $\omega(y) \cap \Lambda^2 = M(y)$. Suppose it is false, then there is a point $u \in \Lambda^2 \cap (\omega(y) \setminus M(y))$. Then $\omega(u) = M(y)$ by Lemma 2.4.4 and Proposition 2.4.1, and $u \in \omega(u)$ since $u \in \Lambda^2 = R$ by Proposition 2.4.3. Thus $u \in M(y)$. This is a contradiction. Therefore $\omega(y) \cap \Lambda^2 = M(y)$. Similarly, we have $\omega(z) \cap \Lambda^2 = M(z)$.

From the second condition of Definition 2.1.3, we know that $\omega(y) \cap \omega(z) \neq \phi$. Let us take $u \in \omega(y) \cap \omega(z)$. Then $M(y) = \omega(u) = M(z)$ by Proposition 2.4.1 and Lemma 2.4.4. That is $\omega(y) \cap \Lambda^2 = M(y) = M(z) = \omega(z) \cap \Lambda^2$. Thus

$$\omega(y) \setminus \omega(z) \subset \Lambda \setminus \Lambda^2.$$

By Proposition 2.4.2, $\Lambda \setminus \Lambda^2$ is countable. So $\omega(y) \setminus \omega(z)$ is countable. This contradicts to the definition of ω-scrambled set. Therefore f has positive entropy. ∎

PROPOSITION 2.4.7 [Cop]. *If f has positive entropy then there exists a closed set $X \subseteq I$ and $m > 0$ such that $f^m(X) = X$ and $f^m|_X$ is at most two to one semiconju-*

gate to the one-sided shift map σ. Furthermore, there are only countably many points in Σ which have 2 preimages.

Since the last statement of this proposition is not included in [Cop], we give some indication of the proof. The idea of the proof of Proposition 2.4.7 given by Coppel [Cop] is as follows. We first prove that if f has positive entropy then $g = f^m$ is strictly turbulent for some $m > 0$, i.e. there are two disjoint closed intervals I_0 and I_1 of I such that $I_0 \cup I_1 \subset g(I_0) \cap g(I_1)$. Then for a finite sequence $s_0 s_1 \dots s_n$ of zeros and ones we let $I_{s_0 s_1 \dots s_n}$ be a component of $\{x \in I_{s_0} : \ g(x) \in I_{s_1}, \dots, g^n(x) \in I_{s_n}\}$ with minimal length. Moreover, for an element $s_0 s_1 \dots s_n \dots$ of Σ we define

$$I_{s_0 s_1 \dots s_n \dots} = \bigcap_{n=0}^{\infty} I_{s_0 s_1 \dots s_n}.$$

This set is a closed subset of $I_{s_0 s_1 \dots s_n}$ and is either a point or a closed interval. Certainly, there are at most countably many such intervals. Let X denote the endpoints of $I_{s_0 s_1 \dots s_n \dots}$, and let $h : X \to \Sigma$ map the endpoints of $I_{s_0 s_1 \dots s_n \dots}$ to $s_0 s_1 \dots s_n \dots$. Then h is a semiconjugacy. h maps at most two points to one, and when h does map two points to one, the two points are the endpoints of some $I_{s_0 s_1 \dots s_n \dots}$. It is easy to prove the following remark.

REMARK. *If one endpoint is a periodic point, then the other must also be a periodic point.*

By the above properties we can prove the following theorem.

THEOREM 2.4.8. *Let f be a continuous map of a closed interval I to itself. The following statements are equivalent.*

I) $h(f) > 0$.

II) There is an uncountable ω-scrambled set S of I such that

$$\bigcap_{x \in S} \omega(x, f) \neq \phi.$$

III) *f is ω-chaotic.*

IV) *There is an ω-scrambled set with two points.*

PROOF: *II*) $\Longrightarrow$ *III*) and *III*) $\Longrightarrow$ *IV*) are obvious. *IV*) $\Longrightarrow$ *I*) is proved by Proposition 2.4.6.

Now let us prove that *I*) $\Longrightarrow$ *II*). By Proposition 2.4.7, there is a closed subset X of I and a positive integer m such that $f^m(X) = X$, and $f^m|_X$ is at most two to one semiconjugate to the one-sided shift map σ, and there are at most countably many points in Σ which have two preimages. By Theorem 2.2.9, σ satisfies statement *II*, so by Theorem 2.3.2, $f^m|_X$ satisfies statement *II*. Let $S(\sigma)$ be the ω-scrambled set constructed in Theorem 2.2.9, and $S(f^m)$ be the ω-scrambled set constructed in Theorem 2.3.2. Then for each $s \in S(\sigma)$, $\omega(s,\sigma)$ contains only one infinite minimal set, and there are only countably many points of $\omega(s,\sigma)$ not in this minimal set. Since $h^{-1}(M)$ is a closed invariant set (as $h(f^m(h^{-1}(M))) = \sigma(h(h^{-1}(M))) = \sigma(M) = M$ implies that $f^m(h^{-1}(M)) \subset h^{-1}(M)$), $h^{-1}(M)$ must contain a minimal set, say $\widetilde{M}$. Then h maps $\widetilde{M}$ onto M since M is a minimal set. Also, $\widetilde{M}$ is infinite since M is. Because there are only countably many points in Σ which have two preimages and $\omega(s,\sigma) \setminus M$ is countable, we have that $\omega(x, f^m) \setminus \widetilde{M}$ is also countable. So $\omega(x, f^m)$ ($x \in S(f^m)$) also contains exactly one infinite minimal set, and, besides the points in the infinite minimal set, there are only countably many points in $\omega(x, f^m)$.

By Theorem 2.3.5, we know that f satisfies statement *II*. ∎

§2.5. Positive Entropy and the Definition of Chaos Given by Devaney

PROPOSITION 2.5.1 [Dev]. *The one-sided shift map is chaotic in the sense of Devaney.*

PROPOSITION 2.5.2 [GotH]. *Let X be a complete, separable metric space. Let $f : X \to X$ be a continuous map. Then the following two statements are equivalent.*

1) f is transitive on X;

2) f is onto and f has a dense orbit on X.

REMARK. *Since a closed subset of a complete separable metric space is still a complete separable metric space, if there is a closed subset D of X such that f is invariant on D, then Proposition 2.5.2 holds for $f|_D$. Note that a compact metric space is complete and separable. When we work on a compact metric space, we will use Proposition 2.5.2 without mention.*

After we have had these propositions, we can prove the following property.

PROPOSITION 2.5.3. *Let $f : I \to I$ be a continuous map. Let D be a closed invariant subset of I. If $f \in C(2^\infty)$, then f is transitive on D implies that there is only one minimal set in D.*

PROOF: If f is transitive, then by Proposition 2.5.2, f has a dense orbit on D. Let $x \in D$ be a point having a dense orbit on D. That is $D = \overline{Orb(x)} = A \cup \omega(x, f)$, where $A \subset Orb(x) \setminus \omega(x, f)$ is at most countable. Note that A contains a closed invariant set provides that $A \cap \omega(x, f) \neq \phi$. Thus A does not contain any minimal set. By Proposition 2.4.1, $\omega(x, f)$ has only one minimal set. So D has only one minimal set.

∎

We call $g : X \to X$ strictly turbulent if there are two disjoint closed subsets S and T such that

$$S \cup T \subset g(S) \cap g(T).$$

PROPOSITION 2.5.4. *If $f : I \to I$ has positive entropy, then there is a positive integer m such that f^m is chaotic on I in the sense of Devaney.*

PROOF: By Proposition 2.4.7, we know that there exists an integer m and a closed set $X \subset I$ such that $f^m|_X$ is semiconjugate to the one-sided shift map σ by a continuous at most two to one map h. Moreover, there are at most countable points in Σ which have two preimages under h. Let $g = f^m$. By a well known fact that σ

has uncountably many points with a dense orbit, we can take one point in Σ which has only one preimage. Let us denote it by s^* and the preimage by x^*.

Then we claim that the $\overline{Orb(x^*, g)}$ contains at least one preimage of any $s \in \overline{Orb(s^*, \sigma)}$ under h. It is easy to see that we need only prove that the $Orb(x^*, g)$ contains at least one preimage of any $s \in Orb(s^*, \sigma)$. Since $h(g^n(x^*)) = \sigma^n(h(x^*)) = \sigma^n(s^*)$ for every positive integer n, if there is an $s \in Orb(s^*, \sigma)$, then there is an $x \in Orb(x^*, g)$ such that $h(x) = s$.

If s is a periodic point with period n, then $\sigma^n(s) = s$. If $h(x) = s$, then $h(g^n(x)) = \sigma^n(h(x)) = \sigma^n(s) = s$. Hence, if s has two preimages x_1 and x_2, then there are four cases as follows

1) $g^n(x_1) = x_1$ and $g^n(x_2) = x_2$;

2) $g^n(x_1) = x_1$ and $g^n(x_2) = x_1$;

3) $g^n(x_1) = x_2$ and $g^n(x_2) = x_2$;

4) $g^n(x_1) = x_2$ and $g^n(x_2) = x_1$.

When 1) or 4) happens, x is a periodic point with period either n or $2n$. By the remark after Proposition 2.4.7, we know that 2) and 3) can not happen. Thus x is a periodic point with period n or $2n$.

Let $s_k \in \Sigma$ be a sequence of periodic points such that $\lim_{k\to\infty} s_k = s^*$. By the above claims, we know that there is a sequence of periodic points $x_k \in \overline{Orb(x^*)}$ of g such that $h(x_k) = s_k$. $\lim_{k\to\infty} x_k = x^*$ since s^* has one preimage. Then $h(x^*) = s^*$. Thus the periodic points in $\overline{Orb(x^*)}$ are dense in itself. Take $D(g) = \overline{Orb(x^*)}$.

So we have found a closed invariant set $D(g)$ which satisfies the first and the second conditions in the definition of chaos given by Devaney.

Let I_0 and I_1 be the intervals defined in the discussion after the Proposition 2.4.7. If we denote the distance between these two intervals by δ, then this is the δ what we want in the definition of the sensitivity. The reason is that for any point $x \in D(g)$ we can find a neighborhood of x so that no point has the same image as x

under h. By the construction of h, if $h(x) \neq h(y)$ then for some nonnegative integer n, one of the points $g^n(x), g^n(y)$ is in I_0 and the other is in I_1. Thus the distance between them is larger than δ. Hence g has sensitive dependence on initial conditions on $D(g)$.

Therefore, we have proved the proposition. ∎

The following proposition is true for a continuous map on a compact metric space.

PROPOSITION 2.5.5. *If, for some $m > 0$, f^m is chaotic in the sense of Devaney, so is f.*

PROOF: Since f^m is chaotic in the sense of Devaney, there is a closed subset $D(f^m)$ which is the orbit closure of some point in itself. Suppose $D(f^m) = \overline{Orb(x^*, f^m)}$.

By Lemma 2.3.3, it is easy to see that

$$(*) \qquad \overline{Orb(x^*, f)} = \bigcup_{i=0}^{m-1} \overline{Orb(f^i(x^*), f^m)}$$

and

$$(**) \qquad f^i(\overline{Orb(x^*, f^m)}) = \overline{Orb(f^i(x^*), f^m)}$$

for $i = 0, 1, \ldots, m-1$. So, $\overline{Orb(x^*, f)} = \bigcup_{i=0}^{m-1} f^i(D(f^m))$. Let D be $\overline{Orb(x^*, f)}$. Then we claim the set D satisfies the definition of chaos in the sense of Devaney. By the construction, the first condition holds. Also, obviously, the third condition holds.

It remains to show that, *if f has sensitive dependence on initial conditions.* Since f^m has sensitive dependence on initial conditions on $\overline{Orb(x^*, f^m)}$, there exists a $\delta > 0$ such that for each $x \in \overline{Orb(x^*, f^m)}$ there exist two sequences $\{n_k\}$ and $\{y_k\} \subset \overline{Orb(x^*, f^m)}$ such that $\lim_{k\to\infty} y_k = x$ and

$$d(f^{n_k}(y_k), f^{n_k}(x)) > \delta.$$

For any $x' \in \overline{Orb(x^*, f)}$, there is a positive integer i such that $x' \in \overline{Orb(f^i(x^*), f^m)}$ by (*). By (**), there is an $x \in \overline{Orb(x^*, f^m)}$ such that $f^i(x) = x'$. Then there are two sequences $\{n_k\}$ and $\{y_k\} \subset \overline{Orb(x^*, f^m)}$ such that

$$d(f^{n_k - i}(f^i(y_k)), f^{n_k - i}(f^i(x))) = d(f^{n_k}(y_k), f^{n_k}(x)) > \delta.$$

Note that $f^i(y_k) \in \overline{Orb(f^i(x^*), f^m)}$ and $\lim_{k\to\infty} f^i(y_k) = f^i(x) = x'$ since f^i is continuous. Because x' is arbitrarily chosen, f has sensitive dependence on initial conditions.

Then we have proved the proposition. ∎

THEOREM 2.5.6. *Let f be a continuous map of a closed interval I to itself. The following statements are equivalent.*

I) $h(f) > 0$.

V) There is a closed invariant subset D of I which is chaotic in the sense of Devaney.

VI) There is a set D as in V) and an uncountable ω-scrambled set $S \subset D$.

PROOF: *I)* $\Longrightarrow$ *V)* is proved in Proposition 2.5.4 and 2.5.5. *V)* $\Longrightarrow$ *I)* comes from Proposition 2.5.3. *VI)* $\Longrightarrow$ *I)* has been proved in Theorem 1.7.

It remains to show that *I)* $\Longrightarrow$ *VI)*. Suppose $h(f) > 0$. Let $D(f^m)$ be a chaotic set for f^m in the sense of Devaney as in Proposition 2.5.4. By the proof of Proposition 2.5.5, there is a chaotic set for f, $D(f)$, with $D(f^m) \subset D(f)$. Let $S(\sigma)$ and $S(f^m)$ be the $\omega-$scrambled sets for σ and f^m constructed in Theorem 2.2.9 and the proof of Theorem 2.4.8. Let h be the semiconjugacy defined in Proposition 2.4.7. By construction, $S(f^m)$ is the preimage of $S(\sigma)$ under h. We may suppose that h maps $S(f^m)$ onto $S(\sigma)$ bijectively since there are only countably many points in Σ which have two preimages by Proposition 2.4.7. By the first claim in the proof of Proposition 2.5.4, $D(f^m)$ contains at least one preimage of each point in $S(\sigma)$. Hence

$S(f^m) \subset D(f^m)$. Let $S(f)$ be the ω-scrambled set for f constructed as in Theorem 2.3.5. Then

$$S(f) \subset S(f^m) \subset D(f^m) \subset D(f).$$

This completes the proof. ∎

CHAPTER 3

SHIFT MAPS ON THE INVERSE LIMIT SPACES

Let X be a compact metric space and let $f : X \to X$ be continuous. The inverse limit space of the sequence

$$X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \dots$$

is defined to be the set of points $\underline{x} = (x_0, x_1, x_2, \dots)$ satisfying $f(x_{i+1}) = x_i$ with the metric

$$\underline{d}(\underline{x}, \underline{y}) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i}$$

where d is a metric on X. Let $\varprojlim(X, f)$ denote the inverse limit space. The shift map $\sigma_f : \varprojlim(X, f) \to \varprojlim(X, f)$ is defined by $\sigma_f((x_0, x_1, \dots)) = (f(x_0), x_0, x_1, \dots)$. Obviously, σ_f is a homeomorphism and $\sigma_f^{-1}((x_0, x_1, x_2, \dots)) = (x_1, x_2, \dots)$. Without causing confusion, we denote σ_f by σ simply. The projection maps $\pi_i : \varprojlim(X, f) \to X$ are defined by $\pi_i((x_0, x_1, \dots, x_i, \dots)) = x_i$ for $i = 0, 1, \dots$. They are continuous. Note that the metric $\underline{d}$ induces the inverse limit topology. This topology has a basis $\mathcal{B} = \{V : V = \pi_i^{-1}(U)$ for some $i \geq 0$ and some open set U in $X\}$.

In this chapter, we prove the following results about the shift map on the inverse limit space of a compact metric space and a sole bonding map: 1) The chain recurrent set of the shift map equals the inverse limit space of the chain recurrent set of the sole bonding map. Similar results are proved for the recurrent set, and almost periodic set. When the mapping is onto, similar result holds for nonwandering set. When the mapping is onto and $\Lambda(\sigma)$ is closed, similar result holds for ω–limit set. But on the interval the condition "the mapping is onto" is not necessary. An example is given to show that generally the "onto" condition is necessary. 2) The shift map on the inverse

limit space is chaotic in the sense of Devaney if and only if its sole bonding map is chaotic. 3) If the sole bonding map is ω-chaotic, then the shift map on the inverse limit space is ω-chaotic. With a modification of the definition of ω-chaos we show the converse is true. 4) An expanding map under which every point is a nonwandering point on a branched 1-manifold is ω-chaotic and chaotic in the sense of Devaney. 5) a continuous map on a closed and invariant subset of the interval with nonempty interior is transitive then it has SDIC on this set.

At last we give a method of constructing chaotic homeomorphisms on chainable continua, especially on pseudo-arcs. If one can find a transitive map f on the interval $[0, 1]$, then the shift map σ_f is $\omega-$chaotic and chaotic in the sense of Devaney. Since Minc and Transue ([MinT]) have constructed a transitive map on $[0, 1]$ whose inverse limit space is a pseudo-arc, we get an ω-chaotic and chaotic homeomorphism in the sense of Devaney on a pseudo-arc. Note that a transitive map of the interval is an onto map, and in this case the whole interval is a chaotic set. The word "chaotic" is the same as in [Ken] where Kennedy constructed a chaotic homeomorphism on a pseudo-arc by a different method.

§3.1. Recurrence Properties

Let W be a closed subset of X. We say that W is strongly invariant if $f(W) = W$.

LEMMA 3.1.1. *Let W be a closed and strongly invariant subset of X. If there is a closed and strongly invariant subset $\underline{W} \subset \varprojlim(X, f)$ such that $\pi_0(\underline{W}) = W$, then $\underline{W} = \varprojlim(W, f)$.*

PROOF: We know that $\pi_i\sigma^i = \pi_0$. Then $\pi_i(\underline{W}) = \pi_i(\sigma^i(\underline{W})) = \pi_0(\underline{W}) = W$ for all $i \geq 0$. Thus $\underline{W} \subset \varprojlim(W, f)$. Moreover, for any point $\underline{y} \in \varprojlim(W, f)$, we have $\pi_i(\underline{y}) \in W$ for all $i \geq 0$. Since $W = \pi_i(\underline{W})$, there is a point $\underline{z}_i \in \underline{W}$ such that $\pi_i(\underline{z}_i) = \pi_i(\underline{y})$, then $\pi_j(\underline{z}_i) = \pi_j(\underline{y})$ for all $j \leq i$. Since $\underline{W}$ is compact, $\underline{z}_i$ has a limit point, say $\underline{z} \in \underline{W}$. Then $\underline{z} = \underline{y}$. Hence $\underline{y} \in \underline{W}$. This proves the lemma. ∎

The following two lemmas are known.

LEMMA 3.1.2. *If $x \in CR(f)$, then for any $\epsilon > 0$ there is an $\epsilon-$chain in $CR(f)$ from x to itself.*

LEMMA 3.1.3. *$CR(f)$ is closed and $f(CR(f)) = CR(f)$.*

LEMMA 3.1.4. *Suppose $f : X \to X$ is semiconjugate to $g : Y \to Y$ by semiconjugacy $h : X \to Y$. Then $h(\Omega(f)) \subset \Omega(g)$.*

PROOF: If $\Omega(f)$ is empty, then it is obviously true. Suppose $\Omega(f)$ is not empty. Let $x \in \Omega(f)$ and $y = h(x)$.

For any open neighborhood V of y, $h^{-1}(V)$ is an open neighborhood of x. Then there is an integer $n > 0$ such that

$$f^n(h^{-1}(V)) \cap h^{-1}(V) \neq \phi,$$

Certainly

$$h(f^n(h^{-1}(V)) \cap h^{-1}(V)) \neq \phi.$$

But

$$\begin{aligned} h(f^n(h^{-1}(V)) \cap h^{-1}(V)) &\subset h(f^n(h^{-1}(V))) \cap h(h^{-1}(V)) \\ &= g^n(h(h^{-1}(V))) \cap V \\ &= g^n(V) \cap V. \end{aligned}$$

Then $g^n(V) \cap V \neq \phi$. Thus $y \in \Omega(g)$. ∎

We assume that f is a continuous map from the interval [0, 1] itself in the following lemma. Let J be a component of $I \backslash \overline{P}$. We say J is of increasing (decreasing) type if for all $x \in J$ and all $n \geq 0$ whenever $f^n(x) \in J$ we have $f^n(x) > x$ ($f^n(x) < x$).

LEMMA 3.1.5 ([Xio, 1983], [Cop, 1984]). *J is either of increasing type or of decreasing type.*

LEMMA 3.1.6 ([BloC, 1986]). *For any point $x \in \Lambda \setminus \overline{P}$, $\omega(x)$ is an infinite minimal set.*

Let $a = min\{x : x \in \overline{P}\}$ and $b = max\{x : x \in \overline{P}\}$. It is easy to see that $f([a, b]) \supset [a, b]$. For there is a sequence of periodic points p_i with $p_i \to b$. Obviously there exists a sequence of periodic points p_i' such that $f(p_i') = p_i$. Then $p_i' \in [a, b]$. Since $[a, b]$ is compact, there is an accumulation point in $[a, b]$ which is mapped to b. Similarly there is a point in $[a, b]$ which is mapped to a. Thus $f([a, b]) \supset [a, b]$.

LEMMA 3.1.7. *If $x \in \Lambda \cap I \setminus [a, b]$, then there is an integer $k > 0$ such that $f^k(x) \in (a, b)$.*

PROOF: If $a = b$, the result is trivial. Assume $a \neq b$. Note that $I \setminus [a, b]$ has at most two components $[0, a)$ and $(b, 1]$ which are also components of $I \setminus \overline{P}$. By Lemma 3.1.5, each one of them is either of increasing type or of decreasing type. Suppose both of them are of increasing type. Since $x \in \Lambda$, there exist a point $y \in I$ and a sequence of integers $n_i > 0$ with $f^{n_i}(y) \to x$. WLOG we assume $x \in [0, a)$ and $f^{n_i}(y) < x$ for all $i > 0$. We will prove the lemma by contradiction.

Suppose $Orb(x) \subset [0, a] \cup [b, 1]$. We will discuss in three cases.

Case 1: There is an integer $k > 0$ such that $f^k(x) = a$. Then $f^{n_i+k}(y) \to f^k(x) = a$. As i is big enough we have $x < f^{n_i+k}(y) < a$. For $j > i$ big enough so that $n_j - n_i - k > 0$, we have $f^{n_j}(y) = f^{n_j-n_i-k}(f^{n_i+k}(y))$. Since $f^{n_j}(y) < x$ for all $j > 0$, it is a contradiction to the assumption of increasing type.

Case 2: There is an integer $k > 0$ such that $f^k(x) = b$. Then $f^{n_i+k}(y) \to f^k(x) = b$. By the hypothesis that $Orb(x) \subset [0, a] \cup [b, 1]$, we have $f(b) \notin (a, b)$. By Lemma 3.1.6, $f(b) \neq b$. Thus $f(b) = a$. Then $f^{k+1}(x) = a$. Thus, we have the situation of Case 1.

Case 3: $Orb(x) \subset [0, a) \cup (b, 1]$. We want to show that there is an $\omega-$limit point of x on $[0,\ a]$. Suppose this is not true. There must be an integer K such that if $k > K$, we have $f^k(x) \in (b,\ 1]$. Since $(b,\ 1]$ is of increasing type, $f^k(x) < f^{k+1}(x)$. Thus $\{f^k(x)\}$ converges to a point $z \in (b,\ 1]$. Since $x \in \omega(y)$, we have $z \in \omega(y)$. Since $x \neq z$, for each integer $j > 0$, we can choose an increasing sequence of integers $\{m_j\}$ such that $f^{m_j}(y) \in (z - \frac{1}{j}, z + \frac{1}{j})$ and $f^{m_j+1}(y) \notin (z - \frac{1}{j}, z + \frac{1}{j})$. Since $(b,\ 1]$ is of increasing type, $f^{m_j+1}(y) \notin (b,\ 1]$. Since $f|_{[a,\ b]}$ is invariant, $f^{m_j+1}(y) \notin [a,\ b]$. Thus $f^{m_j+1}(y) \in [0,\ a)$. Note that $f^k(x) \in (b,\ 1]$ for all $k > K$ and $f^k(x) \to z$. As j is big enough so that $m_j + 1 > K$, we must have $f((z - \frac{1}{j}, z + \frac{1}{j})) \supset [a,\ b]$. Since $a \neq b$, this is a contradiction to the continuity of f. Therefore there are infinitely many points in $[0,\ a] \cap Orb(x)$. Hence $[0,\ a) \cap \omega(x) \neq \phi$.

Let $z \in [0,\ a] \cap \omega(x)$. Then there exists a sequence of integers $\{l_i\}$ such that $f^{l_i}(x) \to z$ and

$$x < f^{l_1}(x) < f^{l_2}(x) < \cdots < f^{l_i}(x) < \cdots < z.$$

Let $\delta = \frac{z-x}{2}$. By the continuity of f, there is a number $\delta > \tau > 0$ such that for any point $z \in (x - \tau,\ x + \tau)$ we have $f(z) > z - \delta > x$. Since $f^{n_i}(y) \to x$, there is an integer i such that $f^{n_i}(y) \in (x - \tau,\ x + \tau)$. Then $f^{n_i+1}(y) > x$. Note that $f^{n_{i+2}}(y) = f^{n_{i+2}-n_i-1}(f^{n_i+1}(y)) < x$. It is a contradiction to the assumption of increasing type.

Therefore $Orb(x) \not\subseteq [0,\ a] \cup [b,\ 1]$. There must be some integer $k > 0$ such that $f^k(x) \in (a,\ b)$. We have proved the lemma. ∎

Let $\underline{x}, \underline{y} \in \varprojlim(X, f)$. It is easy to see that $\underline{y}$ is an ω-limit point of $\underline{x}$ if and only if there exist a sequence of positive integers $\{n_j\}$ and a sequence of positive numbers $\{\epsilon_j\}$ such that $d(y_i,\ x_{n_j+i}) < \epsilon_j$ for $0 \leq i < m_j$ with $m_j \to \infty$ as $j \to \infty$ where d is a metric on X.

THEOREM 3.1.8. *Let $f : X \to X$ be continuous. Properties 1) – 6) hold.*

1) *Let $x \in X$ and let $\underline{x} \in \varprojlim(X, f)$ satisfy $\underline{x} = (x, x_1, x_2, \dots)$. Then $\omega(\underline{x}, \sigma) = \varprojlim(\omega(x, f), f)$.*

2) *$R(\sigma) = \varprojlim(R(f), f)$.*

3) *$AP(\sigma) = \varprojlim(AP(f), f)$.*

4) *$CR(\sigma) = \varprojlim(CR(f), f)$.*

5) *When f is onto, $\Omega(\sigma) = \varprojlim(\Omega(f), f)$.*

6) *If f is onto and $\Lambda(\sigma)$ is closed, then $\Lambda(\sigma) = \varprojlim(\Lambda(f), f)$.*

Let $f : [0,\ 1] \to [0,\ 1]$ be continuous. Then the following property holds.

7) *The condition "f is onto" in 5) and 6) is not necessary, and $\Lambda(\sigma) = \Omega(\sigma)$.*

PROOF: 1) First proof of 1): Let $x \in \omega(x, f)$. There exists a point $\underline{x} \in \varprojlim(\omega(x, f), f)$ with $\pi_0(\underline{x}) = x$. It is easy to see that $\pi_0(\omega(\underline{x}, \sigma)) \subset \omega(x, f)$. On the other hand, for any $y \in \omega(x, f)$, there exists a sequence of points $f^{n_i}(x)$ approaching y. $\{\sigma^{n_i}(\underline{x})\}$ has an accumulation point since $\omega(\underline{x}, \sigma)$ is compact, say $\underline{y} \in \omega(\underline{x}, \sigma)$. Then $\pi_0(\underline{y}) = y$. Thus $\omega(x, f) = \pi_0(\omega(\underline{x}, \sigma))$. Note that both $\omega(x, f)$ and $\omega(\underline{x}, \sigma)$ are closed and strongly invariant. By Lemma 3.1.1, we know that $\omega(\underline{x}, \sigma_f) = \varprojlim(\omega(x, f), f)$.

Second proof of 1: It is easy to see that $\omega(\underline{x}, \sigma_f) \subset \varprojlim(\omega(x, f), f)$. We need prove that $\omega(\underline{x}, \sigma_f) \supset \varprojlim(\omega(x, f), f)$.

For any $\underline{y} \in \varprojlim(\omega(x, f), f)$, let $\underline{y} = (y_0, y_1, \dots, y_i, \dots)$ with $y_i \in X$ for all integers $i \geq 0$. Then $y_i \in \omega(x, f)$. We want to show that $\underline{y} \in \omega(\underline{x}, \sigma_f)$.

Since $y_i \in \omega(x, f)$, We have a sequence $\{i_n\}_{n=1}^{\infty}$ of positive intergers such that $\lim_{n\to\infty} f^{i_n}(x) = y_i$. Since f is continuous,

$$\lim_{n\to\infty} f^{i_n+k}(x) = f^k(y_i) = y_{i-k}$$

for $0 \leq k \leq i$. For any $\epsilon > 0$, there is an integer $N(i)$ such that, as $n \geq N(i)$, we have $d(f^{i_n+k}(x),\ y_{i-k}) < \frac{\epsilon}{4}$, for $0 \leq k \leq i$.

Since X is a compact metric space, it is bounded. Let $M = \sup\{d : d = d(x,y) \text{ for } x,y \in X\}$. Then

$$\sum_{k=i+1}^{\infty} \frac{d(x_k,\ y_k)}{2^k} \leq \frac{M}{2^i}$$

for $\underline{x} = (x_0, x_1, \ldots, x_i, \ldots)$ and $\underline{y} = (y_0, y_1, \ldots, y_i, \ldots)$ for all $i \geq 0$.

Let $\underline{x}_n^i = (f^{i_n+i}(x), \ldots, f^{i_n}(x), \ldots, f(x), x, x_1, x_2, \ldots)$. Then $\underline{x}_n^i \in Orb(\underline{x}, \sigma_f)$. As $n \geq N(i)$ we have

$$\underline{d}(\underline{x}_n^i,\ \underline{y}) \leq \sum_{k=0}^{i} \frac{d(f^{i_n+k}(x),\ y_{i-k})}{2^{i-k}} + \frac{M}{2^i} < \frac{\epsilon}{2} + \frac{M}{2^i}.$$

Obviously, there is an N such that $\frac{M}{2^i} < \frac{\epsilon}{2}$ vas $i > N$.

Denote $\underline{x}_i = \underline{x}_{N(i)}^i \in Orb(\underline{x}, \sigma_f)$. Then

$$\underline{d}(\underline{x}_i,\ \underline{y}) < \frac{\epsilon}{2} + \frac{M}{2^i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as $i > N$. Thus $\underline{y} \in \omega(\underline{x}, \sigma_f)$.

2) Let $\underline{x} \in R(\sigma)$. Then $\underline{x} \in \omega(\underline{x}, \sigma) = \varprojlim(\omega(x,f), f)$. So $\pi_i(\underline{x}) \in \omega(x,f) = \omega(x_i, f)$. Thus $x_i \in R(f)$. Hence $\underline{x} \in \varprojlim(R(f), f)$.

Let $\underline{x} \in \varprojlim(R(f), f)$ and $x = \pi_0(\underline{x})$. Then $\pi_i(\underline{x}) \in R(f)$ for all integers $i \geq 0$. Hence $\pi_i(\underline{x}) \in \omega(\pi_i(\underline{x}), f) \subset \omega(\pi_0(\underline{x}), f) = \omega(x,f)$. Thus $\underline{x} \in \varprojlim(\omega(x,f), f) = \omega(\underline{x}, \sigma)$ by 1). Therefore $\underline{x} \in R(\sigma)$.

3) Let $\underline{x} \in AP(\sigma)$ and $\pi_0(\underline{x}) = x$. Since $AP(\sigma) \subset R(\sigma) = \varprojlim(R(f), f)$, we have $\pi_i(\underline{x}) \in R(f)$ for all integers $i \geq 0$. We need only show that $\omega(\pi_i(\underline{x}), f)$ is a minimal set. Let $x = \pi_i(\underline{x})$. For any $y \in \omega(x,f)$, there is a point $\underline{y} \in \varprojlim(\omega(x,f), f) = \omega(\underline{x}, \sigma)$ with $\pi_0(\underline{y}) = y$ since $f(\omega(x,f)) = \omega(x,f)$.

By 1), $\omega(\underline{y}, \sigma) = \varprojlim(\omega(y,f), f)$. On the other hand, if $\underline{x} \in AP(\sigma)$ then $\omega(\underline{x}, \sigma)$ is a minimal set. Thus $\omega(\underline{x}, \sigma) \subset AP(\sigma)$. Hence $\omega(\underline{y}, \sigma)$ is also a minimal set. By the definition of minimal set, $\omega(\underline{y}, \sigma) = \omega(\underline{x}, \sigma)$ since $\omega(\underline{y}, \sigma)$ is a closed invariant set.

Therefore $\varprojlim(\omega(x,f),f) = \varprojlim(\omega(y,f),f)$. It follows that $\omega(x,f) = \omega(y,f)$. Since y is arbitrary, $\omega(x,f)$ is a minimal set.

Conversely, if $\underline{x} \in \varprojlim(AP(f),f)$ with $\pi_0(\underline{x}) = x$, then for any $\underline{y} \in \omega(\underline{x},f)$, by a similar argument as above, we have

$$\omega(\underline{y},f) = \varprojlim(\omega(\pi_0(\underline{y}),f),f) = \varprojlim(\omega(x,f),f) = \omega(\underline{x},f).$$

Thus $\underline{x} \in AP(\sigma)$.

4) First we show that $CR(\sigma) \subset \varprojlim(CR(f),f)$. Let $\underline{x} = x_0x_1\ldots x_m\cdots \in CR(\sigma)$. Fix an integer $m \geq$. For any $\epsilon > 0$, there is an $\epsilon/2^m$-chain $\underline{x}^0 = \underline{x}$, $\underline{x}^1$, $\ldots$ $\underline{x}^{n-1}$, $\underline{x}^n = \underline{x}$. Then

$$\frac{d(f(x_m^i),\ x_m^{i+1})}{2^m} = \frac{d(x_{m-1}^i,\ x_m^{i+1})}{2^m} < \underline{d}(\sigma(\underline{x}^i),\ \underline{x}^{i+1}) < \frac{\epsilon}{2^m},$$

i.e., $d(f(x_m^i),\ x_m^{i+1}) < \epsilon$. Hence $x_m^0 = x_m$, x_m^1, $\ldots, x_m^{n-1}$, $x_m^n = x_m$ form an ϵ-chain from x_m to x_m. Thus $x_m \in CR(f)$. Since m is chosen arbitrarily, $x_m \in CR(f)$.

Next we prove that $\varprojlim(CR(f),f) \subset CR(\sigma)$. For any $\epsilon > 0$, choose $m > 0$ so that $\sum_{i=m+1}^{\infty} \frac{M}{2^i} = \frac{M}{2^m} < \frac{\epsilon}{2}$ where $M = sup_{x,y\in X} d(x,\ y)$. Since X is compact, $f : X \to X$ must be uniformly continuous. Then there is a number $0 < \delta < \epsilon/4$ such that for any pair of points $x, y \in X$ as $d(x,\ y) < \delta$ we have $d(f^i(x),\ f^i(y)) < \epsilon/4$ for any $m \geq i > 0$.

Since $x_m \in CR(f)$, there is a δ-chain from x_m to x_m, say $x_m^0 = x_m$, x_m^1, $\ldots$, x_m^n $= x_m$ with $x_m^j \in CR(f)$ by Lemma 3.1.2. Then $d(f(x_m^j),\ x_m^{j+1}) < \delta$ for $j = 0,\ 1,\ \ldots,\ n-1$. By the choice of δ we have $d(f^{i+1}(x_m^j),\ f^i(x_m^{j+1})) < \epsilon/4$. Thus

$$\sum_{i=0}^{m} \frac{d(f^{i+1}(x_m^j),\ f^i(x_m^{j+1}))}{2^{m-i}} < \sum_{i=0}^{m} \frac{\epsilon/4}{2^{m-i}} < 2 \times \frac{\epsilon}{4} = \epsilon/2.$$

Since $f|_{CR(f)}$ is onto by Lemma 3.1.3, for each $j \in \{1,\ 2,\ \ldots,\ n-1\}$, there are points x_{m+1}^j, x_{m+2}^j, $\ldots$ such that

$$\underline{x}^j = f^m(x_m^j)f^{m-1}(x_m^j)\ldots f(x_m^j)x_m^jx_{m+1}^jx_{m+2}^j\cdots$$

is well defined in $\varprojlim(CR(f), f)$. Let $\underline{x}^0 = \underline{x} = \underline{x}^n$. Then $\{\underline{x}^0, \underline{x}^1, \ldots, \underline{x}^{n-1}, \underline{x}^n\}$ is an ϵ–chain since

$$\begin{aligned} \underline{d}(\sigma(\underline{x}^j), \underline{x}^{j+1}) &= \sum_{i=0}^{m} \frac{d(f^{i+1}(x_m^j), f^i(x_m^{j+1}))}{2^{m-i}} + \sum_{i=m+1}^{\infty} \frac{d(x_{i-1}^j, x_i^{j+1})}{2^i} \\ &< \epsilon/2 + M/2^m < \epsilon \end{aligned}$$

for $j = 0, 1, \ldots, n-1$.

5) Note that σ is semiconjugate to f by π_i for any $i \geq 0$. Then $\pi_i(\Omega(\sigma)) \subset \Omega(f)$ by Lemma 3.1.4. Thus $\Omega(\sigma) \subset \varprojlim(\Omega(f), f)$.

Next we prove $\Omega(\sigma) \supset \varprojlim(\Omega(f), f)$. Let $M = sup_{x,y \in X} d(x, y)$ and $\underline{x} = (x_0, x_1, \ldots, x_m, \ldots) \in \varprojlim(\Omega(f), f)$. Pick m such that $\sum_{i=m+1}^{\infty} \frac{M}{2^i} = \frac{M}{2^m} < \frac{\epsilon}{4}$. Since $x_m \in \Omega(f)$, we can pick a point y_m so close to x_m that

$$\sum_{i=0}^{m} \frac{d(x_{m-i}, f^i(y_m))}{2^{m-i}} < \frac{\epsilon}{4} \tag{1}$$

and

$$\sum_{i=0}^{m} \frac{d(x_{m-i}, f^{i+k}(y_m))}{2^{m-i}} < \frac{\epsilon}{4} \tag{2}$$

for some integer $k > 0$.

By our assumption that f is onto, we know that there are points $y_{m+1}, y_{m+2}, \ldots$ and $y'_{m+1}, y'_{m+2}, \ldots$ such that $\underline{y} = (f^m(y_m), f^{m-1}(y_m), \ldots, f(y_m), y_m, y_{m+1}, y_{m+2}, \ldots)$ and $\underline{y}' = (f^{m+k}(y_m), f^{m+k-1}(y_m), \ldots, f^{k+1}(y_m), f^k(y_m), y'_{m+1}, y'_{m+2}, \ldots)$ are well defined in $\varprojlim(X, f)$.

Note that

$$\begin{aligned} \underline{d}(\underline{x}, \underline{y}) &= \sum_{i=0}^{m} \frac{d(x_{m-i}, f^i(y_m))}{2^{m-i}} + \sum_{i=m+1}^{\infty} \frac{d(x_i, y_i)}{2^i} \\ &< \frac{\epsilon}{2} + \sum_{i=m+1}^{\infty} \frac{M}{2^i} < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}, \end{aligned}$$

and similarly, $\underline{d}(\underline{x}, \underline{y}') < \frac{\epsilon}{2}$, and

$$\sigma^k(\underline{y}) = (f^{m+k}(y_m), f^{m+k-1}(y_m), \ldots, f^{k+1}(y_m), f^k(y_m), \ldots, y_m, y_{m+1}, \ldots).$$

Then

$$\begin{aligned}\underline{d}(\sigma^k(\underline{y}), \underline{y}') &= \sum_{i=1}^{\infty} \frac{d((\sigma^k(\underline{y}))_{m+i}, y'_{m+i})}{2^{m+i}} \\ &< \sum_{i=1}^{\infty} \frac{M}{2^{m+i}} < \frac{\epsilon}{4}.\end{aligned}$$

Thus

$$\begin{aligned}\underline{d}(\sigma^k(\underline{y}), \underline{x}) &\leq \underline{d}(\sigma^k(\underline{y}), \underline{y}') + \underline{d}(\underline{y}', \underline{x}) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{4} < \epsilon.\end{aligned}$$

Therefore $\underline{x} \in \Omega(\sigma)$.

6) Let $\Sigma = \varprojlim(x, f)$, then

$$\begin{aligned}\Lambda(\sigma) &= \bigcup_{\underline{x} \in \Sigma} \omega(\underline{x}, \sigma) \\ &= \bigcup_{\underline{x} \in \Sigma} \varprojlim(\omega(\pi_0(\underline{x}), f), f) \\ &\subset \varprojlim(\bigcup_{\underline{x} \in \Sigma} \omega(\pi_0(\underline{x}), f), f) \\ &= \varprojlim(\bigcup_{x \in X} \omega(x, f), f) \\ &= \varprojlim(\Lambda(f), f).\end{aligned}$$

Next let $\underline{x} = (x_0, x_1, x_2, \ldots) \in \varprojlim(\Lambda(f), f)$. Then $x_i \in \Lambda(f)$ for all $i > 0$. Then there is a $y_i \in X$ such that $x_i \in \omega(y_i, f)$. Let $\underline{x}^i = (x_0, x_1, \ldots, x_i, y'_{i+1}, y'_{i+2}, \ldots)$ with $y'_j \in \omega(y_i, f)$ for $j \geq i+1$. This can be done since $\omega(y_i, f) = f(\omega(y_i, f))$. For the same reason, $x_j = f^{i-j}(x_i) \in \omega(y_i, f)$ for $j = 0, 1, \ldots, i$. Since f is onto, there exists a point $\underline{y}_i \in \varprojlim(K, f)$ with $\pi_0(\underline{y}_i) = y_i$. Then $\underline{x}^i \in \varprojlim(\omega(y_i, f), f) = \omega(\underline{y}_i, f) \subset \Lambda(\sigma)$.

Note that $\underline{x}^i \to \underline{x}$ as $i \to \infty$ and $\Lambda(\sigma)$ is closed. Thus we have $\underline{x} \in \Lambda(\sigma)$. Therefore, 6) is proved.

7) Let f be a continuous map on $[0,\ 1]$. In 6) we only use the condition " f is onto" when we choose a point $\underline{y}_i$ in $\varprojlim([0,\ 1], f)$ so that $\pi_0(\underline{y}_i) = y_i$. By Lemma 3.1.7, there is an integer $k > 0$ such that $f^k(x_i) \in (a,\ b)$. Thus there exists an integer $n_j > 0$ such that $f^{n_j}(y_i) \in (a,\ b)$. Then $x \in \omega(y, f) = \omega(f^{n_j}(y_i), f)$. Let us substitute $f^{n_j}(y_i)$ for y_i in the proof of 6). Since $f|_{[a,\ b]}$ is onto, there exists a point $\underline{y}_i \in \varprojlim(X, f)$ with $\pi_0(\underline{y}_i) = f^{n_j}(y_i)$. Then the proof of 6) works in this case. Therefore along with 6) we have $\Lambda(\sigma_f) = \varprojlim(\Lambda(f), f)$.

Since it is known that $\Lambda(f) = \bigcap_{n=0}^{\infty} f^n(\Omega(f))$ [Blok, 1982a, Shar, 1982], we have $\varprojlim(\Lambda(f), f) = \varprojlim(\bigcap_{n=0}^{\infty} f^n(\Omega(f)), f)) = \varprojlim(\Omega(f), f) \supset \Omega(\sigma_f)$. On the other hand, $\Lambda(\sigma_f) \subset \Omega(\sigma_f)$. Therefore $\Lambda(\sigma_f) = \Omega(\sigma_f)$. ∎

COROLLARY 3.1.9. *If a subset M in X is a minimal set, then $\varprojlim(M, f)$ is a minimal set. If a subset $\widetilde{M}$ of $\varprojlim(X, f)$ is a minimal set, then $\pi_0(\widetilde{M})$ is a minimal set in X.*

The proof of Corollary 3.1.9 is straightforward.

Example 3.1.1: On the complex plane, we define

$$C = \{z :\ z = (\frac{1}{t+1} + 1)e^{2\pi t i},\ 0 \le t < \infty\}.$$

Let $X = C \cup S^1$ where S^1 is the unit circle with center 0. Then X is a compact metric space with the relative topology as a subset of R^2.

Let $\{0\} \cup \bigcup_{n=1}^{\infty} \{t :\ t = m - \frac{3}{4} \pm \frac{1}{2^{n+2}}, m \ge n\} = \{t_n : t_0 = 0 < t_1 = \frac{1}{8} < t_2 = \frac{3}{8} < \cdots < t_n < \ldots\}$. We define a map f on X as follows. On C, set $f(z(t_n)) = z(t_{n+1})$ where $z(t_n) = (\frac{1}{t_n+1} + 1)e^{(2\pi t_n)i}$ and linearly define f between two adjacent points, i.e., for $t \in (t_n, t_{n+1})$, $f(t) = \phi \circ L_{t_n} \circ \phi^{-1}(t)$ where $\phi : [0,\ \infty) \to C$ maps t to $z(t)$ and $L_{t_n} = t_{n+1} + \frac{t_{n+2} - t_{n+1}}{t_{n+1} - t_n}(t - t_n)$. Then f is continuous on C.

On S^1 we first define $f(i) = i$. Then we define f on the rest of S^1 as follows. Let $\tau_n = \frac{1}{4} + \frac{1}{2^{n+2}}$ and $s_n = \frac{1}{4} - \frac{1}{2^{n+2}}$ for integers $n \ge 1$. define $f(e^{2\pi i s_n}) = e^{2\pi i s_{n+1}}$ for $n \ge 1$, $f(e^{2\pi i \tau_{n+1}}) = e^{2\pi i \tau_n}$ for $n > 1$ and $f(e^{2\pi i \tau_1}) = e^{2\pi i s_1}$. Define f linearly on the rest of S^1. Then $f|_{S^1}$ is continuous.

We claim that f is continuous on X. We need only check that at any point on S^1, f is continuous in X.

Note that, for $e^{2\pi is} \in S^1$ and $e^{2\pi it} \in C$,if $arg\ s = arg\ t$ ($arg\ \tau = arg\ t$) then $arg\ f(e^{2\pi is}) = arg\ f(z(t))$ ($arg\ f(e^{2\pi i\tau}) = arg\ f(z(t))$). For example, let $s \in [s_n,\ s_{n+1}]$ with $s_n = \frac{1}{4} - \frac{1}{2^{n+2}}$ and $t \in [t_i,\ t_{i+1}]$ with $t_i = m - \frac{3}{4} - \frac{1}{2^{n+2}}$ ($m > n$). Assume $arg\ s = arg\ t$. Then $s - s_n = arg\ s - s_n = arg\ t - t_i = t - t_i$. By a simple computation, we can see that $arg\ f(e^{2\pi is}) = arg\ f(z(t))$.

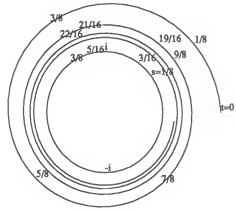

Fig. 1: $(\Gamma \setminus R) \cap (\overline{Orb(T)} - Orb(T)) \neq \phi$.

By using the above fact we have

$$\begin{aligned}
|f(z(t)) - f(e^{2\pi is})| &= |z(t_{i+1} + \frac{t_{i+2} - t_{i+1}}{t_{i+1} - t_i}(t - t_i)) - e^{2\pi i(arg\ f(e^{2\pi is}))}| \\
&= |(\frac{1}{t_{i+1} + \frac{t_{i+2} - t_{i+1}}{t_{i+1} - t_i}(t - t_i) + 1} + 1)e^{2\pi i(arg\ f(z(t)))} \\
&\qquad - e^{2\pi i(arg\ f(e^{2\pi is}))}| \\
&= \frac{1}{t_{i+1} + \frac{t_{i+2} - t_{i+1}}{t_{i+1} - t_i}(t - t_i) + 1} \\
&\leq \frac{1}{t_{i+1} + 1} \leq \frac{1}{m} \leq \frac{1}{n}
\end{aligned}$$

for $arg\ s = arg\ t$. When $arg\ s \neq arg\ t$, by using trianglar inequality and the

continuity of f on C, we can see that $|f(z(t)) - f(e^{2\pi is})|$ is small as $|s-t|$ is small. Thus f is continuous at $e^{2\pi is}$ for $s \in [0, \frac{1}{4})$. Similarly we can prove that f is continuous at $e^{2\pi i\tau}$ for $\tau \in [\frac{1}{4}, 1)$.

Since $f(\phi([m - \frac{3}{4} - \frac{1}{2^{n+3}},\ m - \frac{3}{4} + \frac{1}{2^{n+3}}])) \subset \phi([m - \frac{3}{4} - \frac{1}{2^{n+2}},\ m - \frac{3}{4} + \frac{1}{2^{n+2}}])$ for $m > n$, f is continuous at the point i. Similarly f is also continuous at the point $-i$. Thus f is a continuous map.

Next we show that 5) and 6) fail for the map f. Note that f is not onto.

For any point $e^{2\phi is} \in S^1$, we take a point $x = (1 + \frac{1}{1+t})e^{2\phi it}$ on C for some $t = s + n \in [0,\ +\infty)$. WLOG, we assume that $s \in [0,\ \frac{1}{4}] = [s_0,\ s_1] = [t_0,\ t_1]$. Note that

$$\begin{aligned} f^m(x) &= \phi \circ L_{t_{m-1}} \circ \cdots \circ L_{t_0} \circ \phi^{-1}(x) \\ &= \phi(t_m + \frac{t_{m+1} - t_m}{t_1 - t_0}(\phi^{-1}(x) - t_0)). \end{aligned}$$

If we take m_i so that $arg\ t_{m_i} = arg\ t_0$, then $t_{m_i+1} - t_{m_i} = t_1 - t_0$. Therefore $f^{m_i}(x) = \phi(t_{m_i} - t_0 + \phi^{-1}(x)) = \phi(t_{m_i} - t_0 + s)$ with $t_{m_i} - t_0$ an integer. Thus

$$f^{m_i}(x) = (1 + \frac{1}{1 + t_{m_i} - t_0 + s})e^{2\pi is}.$$

Hence $e^{2\pi is} \in \omega(x, f)$.

Therefore $S^1 \subset \Lambda(f) \subset \Omega(f)$. Obviously, $\Omega(f) \subset S^1$. Thus $S^1 = \Lambda(f) = \Omega(f)$.

Note that $f|_{S^1}$ is a homeomorphism. We have $\varprojlim(\Omega(f), f) = \varprojlim(S^1, f)$ which is homeomorphic to S^1 since $f|_{S^1}$ is a homeomorphism. On the other hand, $\varprojlim(X, f) = \varprojlim(S^1, f)$. Then $f|_{S^1}$ is conjugate to σ_f. Since $\Omega(f|_{S^1}) = \Lambda(f|_{S^1}) = P(f|_{S^1}) = \{-i,\ i\}$, we have $\Omega(\sigma_f) = \Lambda(\sigma_f) = \{-i,\ i\}$. Therefore, $\Omega(\sigma_f) \neq \varprojlim(\Omega(f), f)$ and $\Lambda(\sigma_f) \neq \varprojlim(\Lambda(f), f)$.

§3.2. Chaotic Behaviors

THEOREM 3.2.1. *If f is $\omega-$chaotic, then $\sigma : \varprojlim(X,f) \to \varprojlim(X,f)$ is $\omega-$chaotic.*

PROOF: Let $S(f)$ be an uncountable ω-scrambled set of f on X. Let $x \in S(f)$. Let $S(\sigma)$ consist of one representative of $\pi_0^{-1}(x)$ for each $x \in S(f)$. Then $S(\sigma)$ is obviously uncountable. We want to prove that $S(\sigma)$ is an ω-scrambled set under σ.

Let $\underline{y_i} \in S(\sigma)$ with $i \in \{1,2\}$ and set $x_i = \pi_0(\underline{y_i})$. Then $\omega(\underline{y_i},\sigma) = \varprojlim(\omega(x_i,f),f)$ by 1) of Theorem 3.1.8. It follows that

$$\bigcap_{i=1}^{2} \omega(\underline{y_i},\sigma) = \varprojlim(\bigcap_{i=1}^{2} \omega(x_i,f),f).$$

By the well known fact that any closed and invariant set of a compact metric space contains a minimal set, $\bigcap_{i=1}^{2}\omega(x_i,f)$ Let us denote it by M. Since $f(M) = M$, $\varprojlim(M,f) \neq \phi$. Also it is obvious that $\varprojlim(M,f) \subset \varprojlim(\bigcap_{i=1}^{2} \omega(x_i,f),f)$. Thus $\bigcap_{i=1}^{2}\omega(\underline{y_i},\sigma) \neq \phi$, the second condition of Definition 2.1.1 is satisfied by σ and $S(\sigma)$.

The third condition is obviously satisfied for σ on $S(\sigma)$ by 1) of Theorem 3.1.8.

It is easy to check that $\pi_0(\omega(\underline{y_1},\sigma) \setminus \omega(\underline{y_2},\sigma)) \supset \omega(x_1,\sigma) \setminus \omega(x_2,\sigma)$. If $\omega(x_1,\sigma) \setminus \omega(x_2,\sigma)$ is uncountable, then obviously $\pi_0(\omega(\underline{y_1},\sigma) \setminus \omega(\underline{y_2},\sigma))$ is uncountable. Thus the first condition of Definition 2.1.1 also holds for σ on $S(\sigma)$. ∎

We cannot prove the converse of this theorem. But we can prove that if we replace the first condition of Definition 2.1.1 by the condition:

(*) $\omega(x) \setminus \omega(y)$ contains an infinite minimal set

then the converse of Theorem 3.2.1 is true. This is given as a corollary below. If a map satisfies the condition (*) and the second and third conditions of $\omega-$chaos, then we say that this map is *ω-chaotic with condition (*)*.

COROLLARY 3.2.2. *f is ω-chaotic with condition $(*)$ if and only if σ_f is ω-chaotic with condition $(*)$.*

PROOF: Suppose f is ω-chaotic with condition $(*)$. Let $S(f)$ be an uncountable set satisfying conditions 2) and 3) of Definition 2.1.1 and condition $(*)$. Let $S(\sigma)$ consist of one representative of $\pi_0^{-1}(x)$ for each $x \in S(f)$. Then $S(\sigma)$ is uncountable. By the proof of Theorem 3.2.1, the second and third conditions of Definition 2.1.1 are satisfied by σ and $S(\sigma)$. We need only prove the condition $(*)$.

Let $x_i \in S(f)$ for $i = 1, 2$. Then there is an infinite minimal set $M \subset \omega(x_1, f) \setminus \omega(x_2, f)$. Let $M' = \varprojlim(M, f)$. By Corollary 3.1.9, M' is an infinite minimal set. We claim that $M' \subset \omega(\underline{y_1}, \sigma) \setminus \omega(\underline{y_2}, \sigma)$. First $M' \subset \omega(\underline{y_1}, \sigma) = \varprojlim(\omega(x_1, f), \sigma)$ since $M \subset \omega(x_1, f)$. If $M' \cap \omega(\underline{y_2}, \sigma) \neq \phi$, we can take $z \in M' \cap \omega(\underline{y_2}, \sigma)$. Then $\pi_0(z) \in M$. But $\pi_0(z) \in \pi_0(\omega(\underline{y_2}, \sigma)) \subset \omega(x_2, f)$ too. Thus $M \cap \omega(x_2, f) \neq \phi$. This contradicts the choice of M. So $M' \subset \omega(\underline{y_1}, \sigma) \setminus \omega(\underline{y_2}, \sigma)$.

Suppose σ is ω-chaotic with condition $(*)$. Let $S(\sigma)$ be an uncountable ω-scrambled set. If $\underline{x}$, $\underline{y} \in S(\sigma)$ and $\underline{x} \neq \underline{y}$, then $\pi_0(\underline{x}) \neq \pi_0(\underline{y})$. So $\pi_0(S(\sigma))$ is uncountable. Let $S(f) = \pi_0(S(\sigma))$. Let $x_1 = \pi_0(\underline{y_1})$ and $x_2 = \pi_0(\underline{y_2})$ for $\underline{y_1}$ and $\underline{y_2}$ in $S(\sigma)$. By (1), we know $\pi_0(\bigcap_{i=1}^2 \omega(\underline{y_i}, \sigma)) \subset \bigcap_{i=1}^2 \omega(x_i, f)$. Thus $\bigcap_{i=1}^2 \omega(x_i, f)$ is nonempty since $\pi_0(\bigcap_{i=1}^2 \omega(\underline{y_i}, \sigma))$ is nonempty. That means that the second condition of ω-chaos holds for f. The third condition obviously holds by 1) of Theorem 3.1.8.

Let M be an infinite minimal set contained in $\omega(\underline{y_1}, \sigma) \setminus \omega(\underline{y_2}, \sigma)$. Then $\pi_0(M)$ is an infinite minimal set. Let $\widetilde{M} = \pi_0(M)$. Then $M = \varprojlim(\widetilde{M}, f)$. First, we have $\pi_0(M) \subset \pi_0(\omega(\underline{y_1}, \sigma)) \subset \omega(x_1, f)$. Second, if $\widetilde{M} \cap \omega(x_2, f) \neq \phi$, then $\widetilde{M} \subset \omega(x_2, f)$. We have $M = \varprojlim(\widetilde{M}, f) \subset \varprojlim(\omega(x_2, f), \sigma) = \omega(\underline{y_2}, \sigma)$. But $M \bigcap \omega(\underline{y_2}, \sigma) = \phi$. This is a contradiction. Thus we have $\widetilde{M} \subset \omega(x_1, f) \setminus \omega(x_2, f)$.

Therefore the corollary holds. ∎

By checking the proof of Theorem 2.4.8, we know that if f is ω-chaotic on the interval, then f is ω-chaotic with condition $(*)$. Thus f is ω-chaotic iff f is ω-chaotic with condition $(*)$ on the interval.

REMARK. *In Theorem 3.2.1 and Corollary 3.2.2, if we substitute "is chaotic" or "is chaotic with condition $(*)$" by " satisfies statement II", then they also hold.*

LEMMA 3.2.3 ([Blo, 1975]). *For any continuous map f on a compact metric space, we have $\overline{P(\sigma_f)} = \varprojlim(\overline{P(f)}, f)$.*

PROPOSITION 3.2.4. *A continuous map f of a compact metric space onto itself has SDIC if and only if σ has SDIC.*

PROOF: Suppose σ has SDIC. Then there is a positive small number ϵ such that, for any $\underline{x} \in \varprojlim(X, f)$ and any neighborhood $N(\underline{x})$, there is a point $\underline{y} \in N(\underline{x})$ and an integer $n > 0$ such that $\underline{d}(\sigma^n(\underline{x}),\ \sigma^n(\underline{y})) > \epsilon$.

For any $x \in X$ and any neighborhood $N(x)$, let $\underline{x} \in \pi_0^{-1}(x)$ with $\pi_0(\underline{x}) = x$, let $N(\underline{x})$ be the intersection of $\pi_0^{-1}(N(x))$ and open ϵ–ball about $\underline{x}$. Then $N(\underline{x})$ is an open neighborhood of $\underline{x}$. Since σ has SDIC, there is a point $\underline{y} \in N(\underline{x})$ and an integer $n > 0$ such that $\underline{d}(\sigma^n(\underline{x}),\ \sigma^n(\underline{y})) > \epsilon$. We take n to be the smallest positive integer satisfying this inequality. Then for $0 \leq i < n$ we have $\underline{d}(\sigma^i(\underline{x}),\ \sigma^i(\underline{y})) < \epsilon$. Denote $y = \pi_0(\underline{y})$. Then $y \in N(x)$. By the definition of $\underline{d}$, we have

$$\begin{aligned}\underline{d}(\sigma^n(\underline{x}),\ \sigma^n(\underline{y})) &= d(f^n(x),\ f^n(y)) + \frac{1}{2}\underline{d}(\sigma^{n-1}(\underline{x}),\ \sigma^{n-1}(\underline{y})) \\ &< d(f^n(x),\ f^n(y)) + \frac{1}{2}\epsilon.\end{aligned}$$

Then $d(f^n(x),\ f^n(y)) > \frac{1}{2}\epsilon$. Therefore f has SDIC.

Next we suppose that σ has no SDIC, and show that f has no SDIC.

By the definition of SDIC, we know that, for any $\epsilon > 0$, there is a point $\underline{x} \in \varprojlim(X, f)$ and an open neighborhood $N(\underline{x})$ so that for any $\underline{y} \in N(\underline{x})$ and for any

$n > 0$, we have $\underline{d}(\sigma^n(\underline{x}),\ \sigma^n(\underline{y})) < \epsilon$. Since $\mathcal{B}$ is a base of $\varprojlim(X, f)$, there must be an open subset U of X such that $\underline{x} \in \pi_N^{-1}(U) \subset N(\underline{x})$ for some integer $N > 0$. There is an open subset $V \subset U$ containing x_N such that for any $y \in V$ we have

$$d(f^i(y), f^i(x_N)) < \epsilon \tag{2.1}$$

for $0 \leq i \leq N$.

Note that for any $y \in V \subset U$ there is a $\underline{y} \in \pi_n^{-1}(V) \subset \pi_n^{-1}(U) \subset N(\underline{x})$ such that the $(n+1)$–th coordinate of $\underline{y}$ is y. Set $\underline{y} = (y_0, y_1, ..., y_n, ...)$. Then $y_n = y$. By the fact that σ has no SDIC we know that $\underline{d}(\sigma^m(\underline{x}), \sigma^m(\underline{y})) < \epsilon$ for all integers $m > 0$. Then for $j > N$, we have

$$\begin{aligned} d(f^j(y), f^j(x_N)) &= d(f^{j-N}(y_0), f^{j-N}(x_0)) \\ &< \underline{d}(\sigma^{j-N}(\underline{y}), \sigma^{j-N}(\underline{x})) < \epsilon. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) we see that f has no SDIC. ∎

THEOREM 3.2.5. *σ_f is chaotic in the sense of Devaney iff f is. Furthermore, D is a chaotic set if and only if $\varprojlim(D, f)$ is a chaotic set.*

PROOF: Note that, for a closed and strongly invariant subset $W \subset X$ and some point $x \in W$, $Orb(x, f)$ is dense in W is the same as saying $\omega(x, f) = W$.

Suppose D is a chaotic set of f. Then D is a closed, strongly invariant set. Let $D(\sigma) = \varprojlim(D, f)$. Since $\omega(\underline{x}, \sigma) = \varprojlim(\omega(x, f), f)$ for $x \in D$ by Theorem 3.1.8, if $Orb(x, f)$ is dense in D, then $\omega(\underline{x}, \sigma) = \varprojlim(D, f)$. Thus $orb(\underline{x}, \sigma)$ is dense in $D(\sigma)$. By Lemma 3.2.3, $P(f)$ is dense in D implies that $P(\sigma_f)$ is dense in $D(\sigma) = \varprojlim(D, f)$. By Proposition 3.2.4, σ has SDIC on $D(\sigma)$ since f has SDIC on D. Thus $D(\sigma)$ is a chaotic set. If $D = X$ then $D(\sigma) = \varprojlim(D, f) = \varprojlim(X, f)$.

Conversely, suppose that σ is chaotic. Then there is a chaotic set $D(\sigma)$. It follows that $\sigma(D(\sigma)) = D(\sigma)$. Let $D = \pi_0(D(\sigma))$. Obviously D is closed. Because

$f(D) = f(\pi_0(D(\sigma)) = \pi_0(\sigma(D(\sigma))) = \pi_0(D(\sigma)) = D$, D is strongly invariant under f. By Lemma 3.1.1, we have $D(\sigma) = \varprojlim(D, f)$. We will show that D is a chaotic set for f. By Proposition 3.2.4, f has SDIC on D since σ has SDIC on $D(\sigma)$.

There is a point $\underline{x} \in D(\sigma)$ such that $\omega(\underline{x}, \sigma) = D(\sigma)$. Then $\varprojlim(\omega(\pi_0(\underline{x}), f), f) = \varprojlim(D, f)$. Thus $\omega(\pi_0(\underline{x}), f) = D$. By Lemma 3.2.3, it is easy to see that if $P(\sigma_f)$ is dense in $\varprojlim(D, f)$, then $P(f)$ is dense in D. Therefore, D is a chaotic set of f. If $D(\sigma) = \varprojlim(D, f) = \varprojlim(X, f)$, then $D = \pi_0(D(\sigma)) = X$. ∎

COROLLARY 3.2.6. *Let $f : I \to I$ be a continuous map on the interval. The following are equivalent.*

1) σ_f is chaotic in the sense of Devaney.

2) σ_f is ω-chaotic with condition (*).

3) f has positive topological entropy.

PROOF: Since, on the interval, f is ω-chaotic if and only if f is ω-chaotic with condition (*), by Theorem 2.4.8 and Corollary 3.2.2, we know that f has positive entropy if and only if σ_f is ω-chaotic with condition (*). By Theorem 3.2.5 and Theorem 2.5.6 we know that f has positive entropy if and only if σ_f is chaotic in the sense of Devaney. ∎

§3.3. Expanding Maps on Branched 1-Manifolds

Our next theorem concerns branched 1-manifolds studied by Williams [Wil, 1967]. The definition of a branched 1-manifold is analogous to the definition of a 1-manifold. The only difference is that for a branched 1-manifold, there are three types of coordinate neighborhoods. These are the real line R, $H = \{x \in R : x \leq 0\}$ and $Y = \{(x, y) \in R^2 : y = 0 \text{ or } y = \phi(x)\}$. Here $\phi : R \to R$ is a fixed C^∞ function such that $\phi(x) = 0$ for $x \leq 0$ and $\phi(x) > 0$ for $x > 0$. Let K be a branched 1-manifold. The tangent bundle of K, $T(K)$, may be defined. A differential map $g : K_1 \to K_2$ of branched 1-manifolds induces a map $Dg : T(K_1) \to T(K_2)$ of their tangent bundles.

We say g is an *immersion* if Dg is a monomorphism on the tangent space at each point.

DEFINITION 3.3.1. *If $g : K \to K$ is a C^r immersion of a branched C^r manifold $(r \geq 1)$, then g is expanding relative to a Riemannian metric $\| \ \|$ on $T(K)$, if there are constants $C > 0$, $\lambda > 1$, such that*

$$\|(Dg)^n(v)\| \geq C\lambda^n\|v\|$$

for all $n \in Z^+$, $v \in T(K)$.

Denote $\Sigma = \varprojlim(K, g)$. Then Σ is called a *generalized solenoid* and σ the *shift map determined* by $g : K \to K$.

Let $g : K \to K$ be an immersion of the branched 1-manifold K. In [Wil, 1967], Williams has given the following properties.

THEOREM 3.3.2 ([Wil, 1967]). *If g is expanding and $\Omega(g) = K$, then the periodic points of g are dense in K and the periodic points of σ_g are dense in $\varprojlim(K, g)$.*

The following lemma is known.

LEMMA 3.3.3. *If I and J are closed intervals and $f(I) \supset J$, then there is a closed subinterval $I' \subset I$ such that $f(I') = J$.*

Let $\Sigma_2 = \{0, 1\}^\infty$ with the topology induced by the metric

$$d(s, t) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{2^n}$$

where $s = s_0 s_1 \dots s_n \dots$ and $t = t_0 t_1 \dots t_n \dots$. The map $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(s) = s_1 s_2 \dots s_n \dots$ is called one-sided shift map on Σ_2.

THEOREM 3.3.4. *Let K be a compact branched 1-manifold. Suppose g is expanding and $\Omega(g) = K$. Then g and σ_g are $\omega-$chaotic and chaotic. Furthermore, K is a chaotic set if and only if $\varprojlim(K, g)$ is a chaotic set.*

PROOF: Let I be a closed interval of K. By Theorem 3.3.2, there are two periodic points $a, b \in int(I)$ with $a \neq b$ which are not branch points. Let n be chosen so that g^n fixes a and b. We take a closed neighborhood $N_a \subset int(I)$ of a and a closed neighborhood $N_b \subset int(I)$ of b disjoint from N_a. Since g and so g^n are expanding, there is an integer $m > 0$ such that g^{mn} increases arc length by a factor large enough so that both N_a and N_b covers I under g^{mn}. Denote $f = g^{mn}$. Then $f(N_a) \cap f(N_b) \supset I$. By Lemma 3.3.3, there is a closed interval $N_0 \subset N_a$ and a closed interval $N_1 \subset N_b$ such that $f(N_0) = I = f(N_1)$. Then we know that N_i has disjoint closed intervals N_{i0} and N_{i1} such that $f(N_{ij}) = N_j$ for $i = 0, 1$.

By induction we know that at step l, we can find 2^l disjoint closed intervals N_{α_l} where $\alpha_l = i_1 i_2 \dots i_l$ with $i_k \in \{0, 1\}$ for $k = 1, 2, \dots, l$ such that $f(N_{\alpha_l}) = N_{i_2 \dots i_l}$ with $N_{\alpha_l} \subset N_{i_2 \dots i_l}$.

Let $\alpha = i_1 i_2 \dots i_l \dots$. We claim that $\cap_{l=1}^{\infty} N_{i_1 i_2 \dots i_l}$ is a unique point. This is true because g is expanding iff $f = g^{mn}$ increases arc length by a factor of at least $C\lambda^{nm}$ with $C > 0$ and $\lambda > 1$. We can suppose that m has been chosen big enough so that $C\lambda^{nm} > 1$. Then the arc length of $N_{i_1 i_2 \dots i_l}$ is smaller than $(\frac{1}{C\lambda^{nm}})^l$ times of the arc length of I. Thus the diameter of $N_{i_1 i_2 \dots i_l}$ goes to zero when m goes to infinity. Therefore $\cap_{l=1}^{\infty} N_{i_1 i_2 \dots i_l}$ is a point.

Let $\Lambda = \{x_\alpha : x_\alpha = \cap_{l=1}^{\infty} N_{i_1 i_2 \dots i_l} \text{ for } \alpha = i_1 i_2 \dots i_l \dots \text{ where } i_l = 0, 1\}$. Since Λ is the intersection of all closed intervals, Λ is a closed set. Since

$$f(x_\alpha) = f(\cap_{n=1}^{\infty} N_{i_1 i_2 \dots i_n}) \subset \cap_{n=2}^{\infty} f(N_{i_1 i_2 \dots i_n}) \cap I = \cap_{n=2}^{\infty} N_{i_2 \dots i_n} = x_{i_2 \dots i_n \dots},$$

i.e. $f(x_\alpha) = x_{i_2 \dots i_n \dots} = x_{\sigma(\alpha)}$, Λ is invariant under f.

Define $h : \Lambda \to \Sigma_2$ by $h(x_\alpha) = \alpha$. Then h is a conjugacy because

$$h(f(x_\alpha)) = h(x_{i_2 \ldots i_n \ldots}) = i_2 \ldots i_n \cdots = \sigma(\alpha) = \sigma(h(x_\alpha)).$$

By Theorem 2.2.9 and Proposition 2.5.1, we know that σ is ω-chaotic and chaotic in the sense of Devaney. Thus f is ω-chaotic and chaotic in the sense of Devaney since f is conjugate to σ.

In Σ_2, by the proof of Theorem 2.2.9, there is an ω-scrambled set on which the ω-limit set of each point contains only one infinite minimal set and there are only countably many points outside of the infinite minimal set. Since f is conjugate to σ, f must also have an ω-scrambled set with the same properties. Then by Theorem 2.3.5, g is ω-chaotic.

By Theorem 2.5.5, g is chaotic in the sense of Devaney. By Theorem 3.2.1 and 3.2.5, we have the rest conclusions. ∎

§3.4. Transitivity and Sensitive Dependence

PROPOSITION 3.4.1 ([Blo, 1978, BloC, 1987]). *If $f : I \to I$ is transitive, then the topological entropy of f is at least* $\log \sqrt{2}$.

PROPOSITION 3.4.2 ([BarM, 1985]). *If $f : I \to I$ is transitive, and x is a point whose orbit is dense, then one of the following occurs.*

(i) $Orb(x, f^2)$ is dense in I, in which case $Orb(f^k(x), f^s)$ is dense in I for each $s \geq 1$, $k \geq 0$, or

(ii) $Orb(x, f^2)$ is not dense in I, in which case $I = A \bigcup B$ where $A = \overline{Orb(x, f^2)}$ and $B = \overline{Orb(f(x), f^2)}$ are closed intervals which intersect in a point, and $f(A) = B$, $f(B) = A$. Moreover, for each $s \geq 1$, $Orb(x, f^{2s})$ is dense in A and $Orb(f(x), f^{2s})$ is dense in B.

PROPOSITION 3.4.3 ([BarM, 1985, Nit, 1982]). *If f is transitive, then the set of periodic points of f is dense in I.*

PROPOSITION 3.4.4. *If f is transitive on I, then f has SDIC on I.*

PROOF: Since f is transitive, there is a point $x \in I$ so that $Orb(x, f)$ is dense in I. By Lemma 3.4.2, two cases may happen. If the first case happens, we will prove that f has SDIC on I with $\delta = \frac{1}{10}$. Let $y \in I$. We may assume $y \leq \frac{1}{2}$. The proof for $y > \frac{1}{2}$ is similar. Let N be any neighborhood of y. By Proposition 3.4.3, there is a periodic point p in N of period n with $p < \frac{3}{5}$. Let $g = f^m$. Since $\overline{Orb(x, f)} = I$, there is a point $z \in V \cap Orb(x, f)$. By assumption, $Orb(z, g)$ is dense in I. Then there is an integer j with $g^j(z) > \frac{9}{10}$. Note that $g^j(p) = p < \frac{3}{5}$. We must have either

$$d(g^j(y), g^j(z)) > \frac{1}{10}$$

or

$$d(g^j(y), g^j(p)) > \frac{1}{10}.$$

Hence f has SDIC on I.

If the second case happens, then $Orb(f^{2k}(x), f^{2s})$ is dense in A and $Orb(f^{2k+1}(x), f^{2s})$ is dense in B for each $k \geq 0$ and $s \geq 1$. By the first case, f^2 has sensitive dependence on initial conditions on both A and B. Then both f^2 and f have SDIC on I. ∎

The following lemma is a generalization of Lemma 3.4.2.

LEMMA 3.4.5. *Let $f : I \to I$ be continuous. Let D be a closed invariant set containing an interval. If f is transitive on D, i.e. there is a point x in D so that $\omega(x, f) = D$, then there is a periodic point p in D with period $m > 0$ such that $D = \bigcup_{i=1}^{m} C_i$, where the C_i are closed intervals and pairwise disjoint, and one of the following holds.*

1) $Orb(f^{i-1}(x), f^{sm})$ *is dense in* C_i *for all* $s > 0$, $1 \leq i \leq m$.

2) $C_i = A_i \cup B_i$ *with* $A_i \cap B_i$ *containing only the point* $f^{i-1}(p)$. $Orb(f^{i-1}(x), f^{2sm})$ *is dense in* A_i, *and* $Orb(f^{m+i-1}(x), f^{2sm})$ *is dense in* B_i.

PROOF: Let J be an interval in D. Then $f(J)$ is also an interval by transitivity. Let G be the collection of all nontrivial components of D. Let J_1 and J_2 be two elements in G. Since $\overline{Orb(x,f)} = D$ by assumption, there is a point $z \in J_1 \cap Orb(x,f)$ and an integer $j > 0$ such that $f^j(z) \in J_2$. Since J_2 is a component, and $f^j(J_1)$ is an interval, $f^j(J_1) \subset J_2$. If there is an element J_1' of G such that $f^j(J_1') \subset J_2$, then $J_1' = J_1$. For there are integers l, $s > 0$ such that $f^s(J_1) \subset J_1'$ and $f^l(J_2) \subset J_1$. Then $f^{j+s}(J_1) \subset f^j(J_1') \subset J_2$ and $f^{j+s}(J_1) \subset f^s(J_2)$. Hence $f^s(J_2) \subset J_2$ since J_2 is a component of D. Moreover, $f^{l+s}(J_2) \subset f^l(J_2)$ and $f^{l+s}(J_2) \subset f^s(J_1) \subset J_1'$. Then $f^l(J_2) \subset J_1'$. Thus $J_1 \cap J_1' \neq \phi$. Therefore $J_1 = J_1'$ since both J_1 and J_1' are components of D. Since $\omega(s,f) = D$, D has no trivial components. We can conclude that G is a collection of disjoint closed intervals permuted by f. Thus G is finite. Set $G = \{C_1, C_2, \ldots, C_m\}$. By renumbering, we may assume that $x \in C_1$, $f(x) \in C_2, \ldots, f^{m-1}(x) \in C_m$. Thus $f(C_1) = C_2$, $f(C_2) = C_3, \ldots, f(C_{m-1}) = C_m$, and $f(C_m) = C_1$. Hence f has periodic point p of period m in C_1. Let $g = f^m$. Obviously, $Orb(f^{i-1}(x), g)$ is dense in C_i for each $1 \leq i \leq m$. Since the C_i are intervals, we can use Lemma 3.4.2. If $Orb(x, g^2)$ is dense in C_1, it is easy to see that $Orb(f^{i-1}(x), g^2)$ is dense in C_i. Then 1) follows from the i) in Lemma 3.4.2. If $Orb(x, g^2)$ is not dense in C_1, it is not hard to show that $Orb(f^{i-1}(x), g^2)$ is not dense in C_i either. Then by ii) of Lemma 3.4.2, we can prove 2). ∎

THEOREM 3.4.6. *Suppose that* $D \subset I$ *is closed invariant subset containing an interval. If* $f : I \to I$ *is transitive on* D, *then both* f *and* σ_f *are chaotic in the sense of Devaney with chaotic sets* D *and* $\varprojlim(D, f)$ *respectively, and both* f *and* σ_f *are* $\omega-$*chaotic.*

PROOF: By Lemma 3.4.5, $D = \bigcup_{i=1}^{m} C_i$ with C_i's disjoint closed invariant intervals. And $f^m(C_i) = C_i$. If f is transitive on D, then f^m is transitive on C_i. By Proposition 3.4.3, $\overline{P(f^m)} = C_i$. Then $\overline{P(f)} = D$. By Proposition 3.4.4, f^m has SDIC on C_i. Thus f^m has SDIC on D. Hence f has SDIC on D. Therefore D is a chaotic set for f. By Theorem C, $\varprojlim(D, f)$ is a chaotic set for σ_f. By Proposition 3.4.1, f is $\omega-$chaotic. By Theorem B, σ_f is ω-chaotic. Thus we have proved the theorem. ∎

§3.5. Chaos on Snakelike Continua

Let X be a continuum. We say that X is *chainable* if , for any $\epsilon > 0$, there are finitely many open sets with diameter smaller than ϵ, say O_1, O_2, ..., O_n, which cover X and $O_i \cap O_j \neq \phi$ iff $|i - j| \leq 1$. X is also called *snake-like*. We say that X is *decomposable* if X can be written as the union of two proper subcontinua. If X is not decomposable, then X is *indecomposable*. X is *hereditarily indecomposable* if each of its subcontinua is indecomposable. A *pseudoarc* is a continuum which is homeomorphic to a nondegenerate hereditarily indecomposable chainable continuum.

It is known that the inverse limit spaces of the maps on the interval are chainable. The inverse limit spaces of some maps on the interval can be pseudoarcs [Hed, 1945], [MinT] and etc..

Example 3.5.1: Minc and Transue [MinT] constructed a transitive map on [0, 1] whose inverse limit space is a pseudoarc. By Proposition 3.4.2, the shift map on the pseudoarc induced by the inverse limit space of the transitive map is chaotic in the sense of Devaney with chaotic set $\varprojlim(I, f)$ and $\omega-$chaotic.

Example 3.5.2: The shift map on the pseudoarc constructed by Henderson [Hed] is neither ω-chaotic nor chaotic in the sense of Devaney by Corollary 3.2.2 and Theorem 3.2.5 since the bonding map is not.

Example 3.5.3: Let S^1 be the unit circle. Let $g : S^1 \to S^1$ be defined by $g(z) = z^2$ with $z = x + iy$. Then g is expanding and the nonwandering points

are dense in S^1. By Theorem 3.3.4, g and σ_g are chaotic and $\omega-$chaotic. The space $\varprojlim(S^1, g)$ is usually called the solenoid. This shows that the shift map on the solenoid is chaotic and ω-chaotic.

Example 3.5.4: Let $f : I \to I$ be defined by

$$f_s(x) = \begin{cases} sx & \text{if } x \leq \frac{1}{2}, \\ s(1-x) & \text{if } x > \frac{1}{2}. \end{cases}$$

This map is usually called the tent map. The inverse limit space is a chainable (snake-like) indecomposable continuum. The topological entropy of f_s is $\log s$ if $s \geq 1$. By Theorem 2.4.8 and 2.5.6, if $s > 1$, f_s is chaotic and ω-chaotic. It follows that σ_{f_s} is chaotic and ω-chaotic by Theorems 3.2.1 and 3.2.5.

CHAPTER 4

NONWANDERING POINTS AND TURNING POINTS

Let $I = [0,1]$ be a compact interval on the real line, and let $f : I \to I$ be a continuous map. For each $x \in I$, we let $Orb(x) = \{y \in I : y = f^i(x), i > 0\}$. Let $T = \{x \in I :$ f is not a local homeomorphism at $x\}$ denote the set of turning points of f and $Orb(T) = \{y : y = f^i(x), i > 0, x \in T\}$ denote the set of forward images of turning points of f.

In this chapter we will determine which classes of chain recurrent points (as determined by (0.1)) must be contained in $\overline{Orb(T)}$. By Example 4.3.2, we can see that points in P and $CR \setminus \Omega$ may not be contained in $\overline{Orb(T)}$. Thus we need only consider $\Omega \setminus P$.

For any continuous mapping $f : I \to I$ we will prove that $\Omega(f) \setminus \Gamma(f) \subset \overline{Orb(T)}$, and $\Omega(f) \setminus \Lambda(f) \subset Orb(T)$.

For piecewise monotone maps, Nitecki [Nit, 1980] has proved that $\Lambda \subset \overline{P}$. Thus $\Lambda = \overline{P}$. Xiong [Xio, preprint] has furthermore proved that for any point $x \in \overline{P}$ there is a point $y \in \overline{P}$ such that $x = \omega(y)$. That is $\overline{P} = \Lambda(f|_{\overline{P}})$. In [Xio, 1988], Xiong proves that $\Gamma = \Lambda(f|_{\overline{P}}) = \Lambda(f|_\Lambda) = \Lambda^2$ for any continuous map f. Thus $\Lambda = \overline{P} = \Gamma = \Lambda^2$ for piecewise monotone maps, and $\Omega \setminus \Gamma \subset Orb(T)$. But this fact can not be generalized to any continuous maps. We point out in Example 4.3.1 that the Example 1 in [BloC, 1986] is a counterexample, i.e., there is a point $x \in (\Lambda \setminus \Gamma) \cap (\overline{Orb(T)} \setminus Orb(T))$. We also point out that the point $x \notin \bigcup_{t \in T} \omega(t)$.

In Example 4.3.2, we find out that $\Gamma \setminus P$ need not be a subset of $\overline{Orb(T)}$. In Example 4.3.3, we construct a map so that $(\Gamma \setminus R) \cap (\overline{Orb(T)} \setminus Orb(T)) \neq \phi$. Moreover the point in $(\Gamma \setminus R) \cap (\overline{Orb(T)} \setminus Orb(T))$ is not an ω-limit point of any turning point.

Xiong [Xio, preprint] has proved that if f has no periodic points of periods not a power of 2 and f is piecewise monotone, then any point in $R \setminus P$ is an ω–limit point of some turning point. If we check the proof of this property in [Xio, preprint] carefully, we can see that $R \setminus P \subset \overline{Orb(T)}$ for any continuous map with no periodic points of period not the power of 2. This fact is not true if f has a periodic point with period not the power of 2. Example 4.3.2 is a counterexample.

Without mention, f is always a continuous mapping of I to itself.

§4.1. Nonwandering Points Which Are Not γ-Limit Points

The following lemmas are known.

LEMMA 4.1.1. ([CovN, 1981]) $x \in \Omega$ *if and only if* $x \in \alpha(x)$.

LEMMA 4.1.2. *If* $(f(x) - x)(f(y) - y) < 0$, *then there exists a fixed point between* x *and* y.

LEMMA 4.1.3. *1)* $T(f^n) \subset \bigcup_{i=0}^{n-1} f^{-i}(T(f))$ *(therefore, if* $q \in T(f^n)$, *then there is a point* $q' \in T(f)$ *such that* $f^n(q) \in Orb(q'))$.

2) $\bigcup_{i=0}^{n-1} f^{-i}(T(f) \cup \{0,\ 1\}) \subset T(f^n) \cup \{\epsilon 0,\ 1\}$

PROOF: 1) Given a point $q \in T(f^n)$. We claim that one of q, $f(q)$, ..., $f^{n-1}(q)$ is a turning point of f. Then $q \in \bigcup_{i=0}^{n-1} f^{-i}(T(f))$. Suppose the claim is false. There are D_i $(i = 0, 1, \ldots, n-1)$ such that f is homeomorphic on D_i by the definition of turning points. Choose D_i so that $f(\overline{D_{i-1}}) \subset D_i$. Then $f^n|_{\overline{D_0}} = f \circ f \circ \cdots \circ f|_{\overline{D_0}}$ is a homeomorphism. This contradicts to $q \in T(f^n)$.

2) Let $q \in f^{-i}(T(f)) \setminus \{0, 1\}$. Then $f^i(q) \in T(f)$. We know that there exist two sequences of points $vy_j,\ z_j \to f^i(q)$ with $y_j \neq z_j$ and $f(y_j) = f(z_j)$. Let i be the smallest such integer. Then f^j is a local homeomorphism at q as $j < i$. There are two sequences of different points $y'_j,\ z'_j \to q$ with $f^i(y'_j) = y_j$ and $f^i(z'_j) = z_j$. Thus $f^n(y'_j) = f^n(z'_j)$. Hence $q \in T(f^n)$. Therefore $f^{-i}(T(f)) \subset T(f^n) \cup \{0,\ 1\}$.

Let $q \in f^{-i}(\{0, 1\})$. Then $f^i(q) = 0$ or 1. WLOG assume $f^i(q) = 0$. If $q \in (0, 1)$, then there is a neighborhood U of q such that $f^i(y) \geq 0 = f^i(q)$ for all $y \in U$. Thus $q \in T(f^i) \subset T(f^n)$. Hence $f^{-i}(\{0, 1\}) \subset T(f^n) \cup \{0, 1\}$. ∎

THEOREM 4.1.4. $\Omega(f) \setminus \Gamma(f) \subset \overline{Orb(T)}$.

PROOF: Suppose that $x \in \Omega \setminus \overline{P}$ and C is a connected component of $I \setminus \overline{P}$ which contains x. Obviously, $x \notin R$ since $\overline{P} = \overline{R} \supset R$ [CovH], [Youg]. Then, there exist two points $a, b \in C$ such that $(a, b) \cap Orb(x) = \phi$.

By Lemma 4.1.1, without loss of generality, there exist a sequence of points $y_i \in (a, x)$ and a sequence of positive integers $n_i \to \infty$ such that $f^{n_i}(y_i) = x$ and $y_i \to x$ as $i \to \infty$. Obviously, we could suppose that $y_i < y_{i+1}$ and $n_i < n_{i+1}$.

Since $f^{n_i}(y_i) = x > y_i$ and $(a, b) \cap P = \phi$, it follows that $f^{n_i}(y_{i-1}) > y_{i-1}$ by Lemma 4.1.2. Also $f^{n_i}(y_{i-1}) = f^{n_i - n_{i-1}}(f^{n_{i-1}}(y_{i-1})) = f^{n_i - n_{i-1}}(x)$. Since $(a, b) \cap Orb(x) = \phi$, it follows that $f^{n_i}(y_{i-1}) \geq b$. By the same reason, $f^{n_i}(x) \geq b$.

Since $f^{n_i}(y_i) = x < b$, there must be a turning point $q \in (y_{i-1}, x)$ of f^{n_i} with $q < f^{n_i}(q) \leq x$. That is to say that $f^{n_i}(q) \in (y_{i-1}, x]$. By Lemma 4.1.3, there exists a point $q' \in T(f)$ such that $f^{n_i}(q) \in Orb(q')$.

Since $f^{n_i}(q) \to x$ as $i \to \infty$ it follows that $x \in \overline{Orb(T(f))}$.

Thus $\Omega \setminus \overline{P} \subset \overline{Orb(T(f))}$. Note that the proof of Theorem 1 in paper [Xio, preprint] implies that $\overline{P} \setminus \Gamma \subset \overline{Orb(T(f))}$. Therefore we have proved the theorem. ∎

§4.2. Nonwandering Points Which Are Not ω-Limit Points

LEMMA 4.2.1. *If K_i $(i = 1, 2, \ldots)$ are closed intervals such that $K_{i+1} \subset K_i$ and $f^{n_i}(K_i) \supset K_{i+1}$ for all $i \geq 1$, and $\cap_{i=1}^{\infty} K_i$ consists of a single point x, then there exists a point $y \in K_1$ such that $f^{l_j}(y) \in K_{j+1}$ and $f^{l_j}(y) \to x$ as $j \to \infty$ with $l_j = \sum_{i=1}^{j} n_i$.*

PROOF: By assumption, we have $f^{l_i}(K_1) \supset K_{i+1}$ for all $i \geq 1$. We can form a sequence of closed intervals $\{K'_j\}$ such that $K'_1 \subset K_1, K'_{j+1} \subset K'_j$, and $f^{l_j}(K'_j) =$

K_{j+1} for each positive integer j. It follows that $f^{l_i}(K'_j) \subset K_{i+1}$ for all $i < j$. Let $y \in \bigcap_{j=1}^{\infty} K'_j$. Then $f^{l_j}(y) \in K_{j+1}$ for all $j \geq 1$ and furthermore $f^{l_j}(y) \to x$. ∎

If C is a component of $I \setminus \overline{P}$, it has been shown [Xio, 1983b], [Cop, 1984] that for any $x, y \in C$,

$$(f^n(x) - x)(f^m(y) - y) > 0$$

for all integers $n, m > 0$ such that $f^n(x), f^m(y) \in C$.

If $f^n(x) > x$ whenever $f^n(x) \in C$ (or $f^n(x) < x$), we say that C is of increasing type (or decreasing type).

LEMMA 4.2.2. *If $x \notin \Lambda$ is in a component C of $I \setminus \overline{P}$ of increasing type, then there is an interval $[z, x]$ such that, for any $y \in [z, x]$ and any positive integer n, $f^n(y) \notin [z, x)$.*

PROOF: Suppose the conclusion is not true. Then there is an increasing sequence of points $y_i \in C$ with $y_i \neq x$ and $y_i \to x$, and a sequence of integers $n_i > 0$ such that $y_i < f^{n_i}(y_i) < y_{i+1} < x$. Since $x \notin \Lambda$ and $\Lambda \supset R$, we have $x \notin R$. Since $f^{n_i}(y_i) > y_i$ and there is no periodic point in $[y_i, x]$, we know that $f^{n_i}(x) > x$ by Lemma 4.1.2.

Denote $K_i = [y_i, x]$. Then

$$f^{n_i}(K_i) \supset [f^{n_i}(y_i), f^{n_i}(x)] \supset K_{i+1}.$$

Also, $\bigcap_{i=1}^{\infty} K_i$ consists of a single point x. By Lemma 4.2.1, there is a point $y \in K_1$, so that $f^{l_j}(y) \to x$ where $l_j = \Sigma_{i=1}^{j} n_i$. It follows that $x \in \omega(y) \subset \Lambda$. This contradicts the assumption that $x \notin \Lambda$. ∎

THEOREM 4.2.3. $\Omega(f) \setminus \Lambda(f) \subset Orb(T)$.

PROOF: Let $x \in \Omega \setminus \Lambda$. Since Λ is closed, by Theorem 4.1.4, there is an interval (a, b) containing no points of Λ such that $x \in (a, b)$ and $(a, b) \cap Orb(x) = \phi$. Also, as in the proof of Theorem 4.1.4, for any $x \in \Omega \setminus \Lambda$ we may assume without loss of generality that there exist a sequence of points $y_i < x$ in (a, b) and an increasing sequence of

positive integers $n_i \to \infty$ such that $f^{n_i}(y_i) = x$, for all $i = 1, 2, \ldots$ and $y_i \to x$. Then (a, b) is of increasing type. By Lemma 4.2.2, there is an interval $[z, x]$ so that, for any $y \in [z, x]$ and any integer $n > 0$, we have $f^n(y) \notin [z, x)$. Without loss of generality, we can assume that $y_1, y_2, \ldots, y_i, \ldots$ are all in $[z, x]$. Then $f^n(y_i) \notin [z,\ x]$.

By Lemma 4.1.2, for any point w in (y_1, y_3), we have $f^{n_2}(w) \geq x$ since $f^{n_2}(y_2) = x$. Thus y_2 is a turning point of f^{n_2}. By Lemma 4.1.3, $x = f^{n_2}(y_2) \in Orb(T(f))$. ∎

§4.3. Examples

The following lemma is obvious.

LEMMA 4.3.1. *If $\{I_i\}_{i=1}^n$ is a collection of disjoint closed intervals such that $f(I_{i-1}) \supset I_i$ for $i = 2, \ldots, n$, and $f(I_n) \supset I_1$, then f has a periodic point with period n.*

PROPOSITION 4.3.2 [XIO, 1983A]. *$\overline{R(f)} = R(f)$ implies that f has only periodic points of periods 2^n, $n \geq 0$.*

In Example 4.3.1 we show that $\Lambda \setminus \overline{P}$ is not always contained in $Orb(T)$.

Example 4.3.1 ([BloC, 1986]): Let $f : [-1, 2] \to [-1, 2]$ be the map constructed by Block and Coven in Example 1 [BloC, 1986]. The following properties are either already observed by Block and Coven [BloC, 1986] or easy to be seen.

(1) The Cantor middle third set C in $[0, 1]$ is a minimal set under f.

(2) $\omega(-2/3) = C$.

(3) $-2/3 \in \Lambda \setminus \overline{P}$.

(4) Any point $t \in [-2/3,\ 0]$ is a turning point and $f(t) = 2/3 \in C$. 1 is a turning point and $1 \in C$. Let $s_n = \frac{3^n - 1}{3^n}$ for integers $n \geq 1$. Then each s_n is a turning point and $s_n \in C$.

(5) Except for those turning points described in (4), all other turning points can be represented by $t_n = 1 - \frac{1}{2 \times 3^{n-1}}$. Moreover $f^{m_n}(t_n) = -2/3 - 1/3^{n+3}$ for some integer $m_n > 0$ and $f^2(-2/3 - 1/3^n) = 2/9 + 1/3^{n+1} \in C$.

By (5) we know that $-2/3 \in \overline{Orb(T)}$. $-2/3$ can never be the image of a turning point in (4) since $f(C) \subset C$ by (1). For the t_n's in (5), we know that for $0 < j < m_n$ we have $f^j(t_n) \in [0,2]$. Also, $f^{m_n}(t_n) = -2/3 - 1/3^{n+3}$, $f^{m_n+1}(t_n) \in [1/3,\ 2/3]$, and $f^{m_n+2}(t_n) \in C$. Thus $-2/3 \notin Orb(t_n)$. Therefore $-2/3 \notin Orb(T)$. By (3), $-2/3 \in (\Lambda \setminus \overline{P}) \cap (\overline{Orb(T)} \setminus Orb(T))$.

Moreover by (1) and the above argument it is easy to see that $\omega(t) \subset C$ for any turning point $t \in T$. Thus $-2/3 \notin \bigcup_{t \in T} \omega(t)$.

Example 4.3.2: (See Fig. 2) Let $f : [0,\ 3] \to [0,\ 3]$ be defined as follows.

$$f(x) = \begin{cases} \sqrt{x}, & \text{when } x \in [0,1], \\ 4x - 3, & \text{when } x \in [1, 3/2], \\ -2x + 6, & \text{when } x \in [3/2, 3]. \end{cases}$$

For any point $x \in (0,1)$ we have that $1 \in \alpha(x) \cap \omega(x)$. Note that 1 is a fixed point. So $x \in CR$. But $x \notin \Omega$. Thus $x \in CR \setminus \Omega$. Obviously $x \notin \overline{Orb(T)}$.

Next we show that $\Gamma \setminus R$ is not be contained in $\overline{Orb(T)}$.

The turning point of f is $3/2$. $Orb(\{3/2\}) = \{0, 3/2, 3\}$. Since $\frac{3}{2}$ and 3 are wandering points and 0 is a fixed point, we have $\Omega = \Gamma$ by Theorem 4.1.4. Moreover, 0, $3/2$ and 3 are not in the set $\Gamma \setminus R$, i.e., $(\Gamma \setminus R) \cap \overline{Orb(T)} = \phi$. Next we show that $\Gamma \setminus R \neq \phi$. Let $I_1 = [1, 3/2]$ and $I_2 = [3/2, 2]$ and $I_3 = [2, 3]$. Then $f(I_1) \supset I_2$ and $f(I_2) \supset I_3$ and $f(I_3) \supset I_1$. Then f has a periodic point with period 3 by Lemma 4.3.1. Since $\overline{R} \subset \Omega = \Gamma$, it follows from Proposition 4.3.2 that $\Gamma \setminus R \neq \phi$.

Note that $R \neq P$. For otherwise, $\overline{P} = P$ by [Xio, 1986], a contradiction to $\Gamma \setminus R \neq \phi$. By the above argument we know that $\frac{3}{2}$ and 3 are not recurrent points. Thus $(R \setminus P) \cap \overline{Orb(T)} = \phi$.

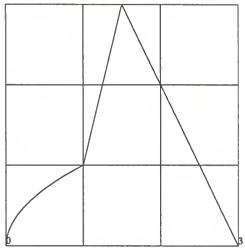

Fig. 2: $P \not\subset \overline{Orb(T)}$. $CR \setminus \Omega \not\subset \overline{Orb(T)}$. $\Gamma \setminus R \cap \overline{Orb(T)} = \phi$.

At last we construct a map on the interval $[0,\ 1]$ so that there is a point belonging to $(\Gamma \setminus R) \cap (\overline{Orb(T)} \setminus Orb(T))$.

Example 4.3.3: We will define f on $[0,\ 1]$ as follows.

Step 1: Set $f(0) = f(1/9) = f(1) = 1$, $f(2/9) = f(1/3 + 2/9) = 2/9$ and $f(2/3) = 0$.

Except on $[1/9,\ 2/9] \cup [2/3,\ 1]$, linearly define f by connecting two neighboring points. Let $p_1^0 = \frac{2}{9}$. p_1^0 is a fixed point.

Step 2: Let $p_2^0 = \frac{1}{9} + \frac{1}{27}$. Then there is a point $p_2^1 \in (1/3 + 2/9,\ 2/3)$ such that $f(p_2^1) = p_2^0$. Define $f(p_2^0) = p_2^1$. Then $\{p_2^0,\ p_2^1\}$ is a periodic orbit of period 2. Define $f(2/3 + 1/9) = f(2/3 + 1/9 + 2/27) = p_2^1$ and $f(2/3 + 2/9) = 2/3$. Linearly extend f to $[0,\ 1/9] \cup [p_2^0,\ 2/3 + 2/9]$.

Suppose we have defined Step $n-1$. Let us define Step n.

Step n: (See Fig. 3) Let $p_n^0 = \frac{1}{9} + \frac{1}{3^{n+1}}$. Then there are points $p_n^1 \in (1/3 + 2/9,\ 2/3)$, $p_n^i \in (2/3 + \cdots + 2/3^{i-1} + 1/3^i + 2/3^{i+1},\ 2/3 + \cdots + 2/3^i)$ such that $f(p_n^{i+1}) = P_n^i$ with $i = 0, 1, \ldots, n-2$. Define $f(p_n^0) = p_n^{n-1}$, $f(2/3 + \cdots + 2/3^{n-1} + 1/3^n) = f(2/3 + \cdots + 2/3^{n-1} + 1/3^n + 2/3^{n+1}) = p_n^{n-1}$ and $f(2/3 + \cdots + 2/3^n) =$

$2/3 + \cdots + 2/3^{n-1}$. Linearly extend f to $[0,\ 1/9] \cup [p_n^0,\ 2/3 + \cdots + 2/3^n]$.

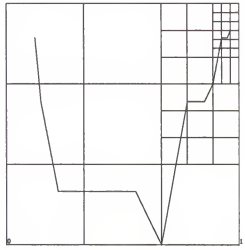

Fig. 3. The graph of f at Step 3. $1/9 \in (\Gamma \setminus R) \cap (\overline{Orb(T)} \setminus Orb(T))$.

The map is continuous since $p_n^0 \to 1/9$ and $\sum_{n=1}^{\infty} 2/3^n = 1$.

(1) Since $f(1/9) = 1$ and 1 is a fixed point, $1/9 \notin R$. Since $p_n^0 \to 1/9$ and p_n^0's are periodic points, we have $1/9 \in \overline{P}$.

(2) $f([p_3^0,\ p_2^0]) = [p_2^1,\ p_3^2]$, $f([p_2^1,\ p_3^2]) \supset [1/9,\ p_4^0]$, and $f^n([1/9,\ p_n^0]) = [0,\ 1] \supset [1/9,\ p_{n+1}^0]$ as $n > 1$. Thus by Lemma 4.2.1, there is a point $x \in [p_3^0,\ p_2^0]$ such that $1/9 \in \omega(x)$.

On the other hand, for every point $x \in (1/9,\ 2/9)$, we have $1/9 \in \alpha(x)$. For there are points $x_n \in (\frac{2}{3} + \cdots + \frac{2}{3^n}, \frac{2}{3} + \cdots + \frac{2}{3^n} + \frac{1}{3^{n+1}})$ such that $f(x_n) = x_{n-1}$ for $n > 1$ and $f(x_1) = x$ with $x_n \to 1$. Also there are points $y_n \in (\frac{1}{9}, \frac{2}{9})$ such that $f(y_n) = x_n$ with $y_n \to \frac{1}{9}$. Thus $\frac{1}{9} \in \alpha(x)$. So $1/9 \in \alpha(x) \cap \omega(x) \subset \Gamma$.

By the construction it is easy to see that $p_n^0 \in Orb(T)$. Thus $\frac{1}{9} \in \overline{Orb(T)}$. But $\frac{1}{9}$ is not in any periodic orbit and is not an $\omega-$limit point of any turning point.

Therefore, $\frac{1}{9} \in (\Gamma \setminus R) \cap (\overline{Orb(T)} \setminus Orb(T))$.

CHAPTER 5

A NOTE ON TYPE 2^∞ MAPS AND UNIMODAL MAPS

Again let $I = [0,1]$, and let $f : I \to I$ be a continuous self-mapping. We say that f is a Type 2^∞ mapping if f does not have any periodic point with period $m2^n$ for $m > 1$ odd and $n \geq 0$.

The forward and backward orbit are defined as follows. For $x \in I$,

$$Orb_+(x) = \{y : y = f^m(x) \in I \text{ for } m > 0\}$$

and

$$Orb_-(x) = \{y \in I : f^m(y) = x \text{ for } m \geq 0\}.$$

Block and Coven [BloC, 1986] have proved that, for a continuous map, an ω-limit point not in $\overline{P}$ has an infinite minimal set as its ω-limit set.

In this chapter, we prove that, for Type 2^∞ maps, a chain recurrent point which is not in $\overline{P}$ has an infinite minimal set as its ω-limit set. An example is given to show that for non-type 2^∞ maps the above result may not be true.

Xiong [Xio, 1984] and Smítál [Smi, 1986] have given examples to show that a type 2^∞ map may be chaotic in the sense of Li-Yorke. In this chapter, we will show that a scrambled set can not be taken from the nonwandering set.

Turning points are as defined in Chapter 3. If a map has only one turning point, then we say that this map is unimodal. Let t be the turning point. If f takes the maximum value at t on I, we say that f is an upper unimodal map. Otherwise f is a lower unimodal map. In the following disscussion, we will concentrate on the

continuous onto unimodal maps. We suppose that, for a continuous map on I, either $f(0) = 1$ and $f(t) = 0$ or $f(1) = 0$ and $f(t) = 1$ where t is the unique turning point.

Note that this is not a special case. In fact, except for possibly some fixed points which are not on $[f(t), f^2(t)]$ or $[f^2(t), f(t)]$, all nonwandering points are on this interval. It is enough to study the dynamical behaviour on this interval in order to study the dynamical behaviour on I. Obviously, f is of Type 2^∞ iff $f|_{[f(t),f^2(t)]}$ or $f|_{[f^2(t),f(t)]}$ is of Type 2^∞.

§5.1. Preliminaries

The following are some known properties.

LEMMA 5.1.1 (Lemma 6 in [Li]). *Any Type 2^∞ unimodal mapping f has the following properties:*

1) I is the union of disjoint closed intervals $\{I_{\sigma_n}\}_{n=1}^\infty$ and open intervals $\{D_{\sigma_{n-1}}\}_{n=1}^\infty$ where $\sigma_n = i_1 i_2 \dots i_n$ and $i_k \in \{0, 1\}$ for $k = 1, 2, \dots, n$ and where D_{σ_n} and I_{σ_n} satisfy

i) $I_{\sigma_{n-1}} \supset I_{\sigma_n}$ if $\sigma_n = \sigma_{n-1} i_n$,

ii) $I_{\sigma_n} = I_{\sigma_n 0} \cup D_{\sigma_n} \cup I_{\sigma_n 1}$,

iii) $\overline{D_{\sigma_n}}$ is disjoint from $\overline{D_{\sigma_m}}$ if $\sigma_n \neq \sigma_m$,

so that $I^n = \bigcup_{\sigma_n} I_{\sigma_n}$ forms a filtration order for $\sigma_n = \sigma_{n-1} i_n$,

$$I^\infty \subset \cdots \subset I^n \subset \cdots \subset I^2 \subset I^1$$

and $I = I^\infty \bigcup D^\infty$ and $I^\infty \bigcap D^\infty = \phi$ where $I^\infty = \bigcap_{n=1}^\infty I^n$ and $D^\infty = \bigcup_{n=1}^\infty D^n$ with $D^n = \bigcup_{\sigma_{n-1}} D_{\sigma_{n-1}}$ and $D^0 = D_{\sigma_0} = D$.

2) I^n is f-invariant and f permutes $I_n = \{I_{\sigma_n}; \sigma_n = i_1 i_2 \dots i_n, i_k \in \{0, 1\}, k = 1, 2, \dots, n\}$.

3) $f^{2^n}|_{I_{\sigma_n}} \in C(2^\infty, I_{\sigma_n})$ *and* $f^{2^n}|_{I_{\sigma_{n-1}0}}$ *is a lower unimodal map and* $f^{2^n}|_{I_{\sigma_{n-1}1}}$ *is an upper unimodal.* $f^{2^n}|_{I_{\sigma_n}}$ *is increasing. Periodic points with period greater than* 2^n *is contained in* $\bigcup_{i\geq n}^{\infty} I_i$.

4) $P(f) \subset D^\infty$ *and* $\overline{P(f)} - P(f) \subset I^\infty$.

Let t *denote the turning point of* f. *Then*

i) $Orb_+(t) \subset I^\infty$,

ii) $\{f^i(t) : i = 1, 2, \ldots, 2^{n-1}\}$ *are the endpoints of* I^n, *and also the endpoints of* $\overline{D^n}$.

iii) $\{y \in I^n : f^i(y) = t, i = 0, 1, 2, \ldots, 2^{n-1}\}$ *are the turning points of* f^{2^n} *on the union of the elements of* I_n.

5) $f^{2^n}|_{I_{\sigma_n}}$ *has a unique fixed point* $\beta_{\sigma_n} \in D_{\sigma_n}$.

Denote

$$A_n = \{\beta_{\sigma_n} : \sigma_n = i_1 i_2 \ldots i_n, i_k \in \{0, 1\}, k = 1, 2, \ldots, n\}$$

and $A = \bigcup_{n=1}^{\infty} A_n$, *then* f *permutes the elements in* A_n. *And there are* $\beta_p, \beta_q \in A \cap (\beta_{\sigma_n}, \beta_{\sigma_n'})$ *where* $p = \sigma_{m_1}$ *and* $q = \sigma_{m_2}$ *with* $m_1 < n < m_2$. *There is a* $\beta_{\sigma_n} \in A \cap (\beta_{\sigma_r}, \beta_{\sigma_s})$ *with* $n \geq max(r, s)$ *for* $s \neq r$.

$5'$) $\bigcup_{i=0}^{\infty} \{y; f^i(y) = t\} \cap (\beta_{\sigma_r}, \beta_{\sigma_s}) \neq \phi$ *for any* $r \neq s$.

The property $5'$ is new. We prove it as follows.

PROOF: By 5), there is an $n > max(r, s)$ and σ_n such that $\beta_{\sigma_n} \in A \cap (\beta_{\sigma_r}, \beta_{\sigma_s})$. Since $\beta_{\sigma_r} \in D_{\sigma_r}$, we have that $I_{\sigma_n} \subset (\beta_{\sigma_r}, \beta_{\sigma_s})$. Then, the turning point of f^{2^n} on I_{σ_n} is in $(\beta_{\sigma_r}, \beta_{\sigma_s})$.

By 4) we know that

$$\bigcup_{i=0}^{\infty} \{y; f^i(y) = t\} \cap (\beta_{\sigma_r}, \beta_{\sigma_s}) \neq \phi.$$

for any $r \neq s$. ∎

LEMMA 5.1.2 ([Li]). *In Lemma A, if $x_{\theta_i} \in D_{\theta_i}$ for $i = 1, 2, \ldots,$ and $\lim_{i\to\infty} x_{\theta_i} = x_0$, then $\lim_{i\to\infty} y_{\theta_i} = x_0$ for any $y_{\theta_i} \in D_{\theta_i}$, where $\{\theta_i\}$ is a sequence of permutations of $\{\sigma_n\}$.*

LEMMA 5.1.3 ([Li], [Xio, preprint]). *If $f \in C(2^\infty)$ is unimodal, then $\overline{R(f)} = R(f)$.*

LEMMA 5.1.4 [Liao, 1985]. *f has a periodic point of period not a power of 2 if and only if there is a chain recurrent point x such that $\omega(x)$ is a periodic orbit but x is not periodic.*

§5.2. Some Properties of Type 2^∞ Maps

In this section, we suppose f is of type 2^∞.

THEOREM 5.2.1. *Any ω-limit set can contain only one point of a scrambled set. The nonwandering set can contain only countably many points of a scrambled set.*

PROOF: Let $z \in I$ and $\omega = \omega(z)$. We are going to show that, for any points $x, y \in \omega$ with $x \neq y$, either $\limsup_{i\to\infty} |f^i(x) - f^i(y)| = 0$ or $\liminf_{i\to\infty} |f^i(x) - f^i(y)| > 0$. Thus x and y can't be in the same scrambled set. Hence ω can contain only one point of a scrambled set.

By

$$\omega \subset Orb(J_k) = \bigcup_{i=1}^{2^k} f^j(J_k)$$

for all $k \geq 0$ where J_k is an f-periodic interval of period 2^k. For any $x, y \in \omega$ with $x \neq y$ with $x \neq y$ and any sequence $\{n_k\}$, if $x, y \in f^{n_k}(J_k)$ for all $k \geq 1$, then $[x, y] \subset \bigcap_{k=1}^{\infty} f^{n_k}(J_k)$ since $f^{n_k}(J_k)$ is an interval. Since $f^{n_k}(J_k)$ is an f-periodic interval, it is disjoint from other intervals in $Orb(J_k)$. Then $\{f^i([x, y])\}_{i=1}^{2^k}$ is a disjoint sequence. As $k \to \infty$, we get an infinite disjoint sequence $\{f^i([s, y])\}_{i=1}^{\infty}$. For any $\epsilon > 0$, there is an integer N, such that $length(f^i([s, y])) < \epsilon$. As $i > N$, since $\bigcup_{i=1}^{\infty} length(f^i([x, y])) < 1$. That is $\limsup_{i\to\infty} |f^i(x) - f^i(y)| = 0$.

If x, y are not always in the same interval, then we can assume that $x \in J_k$ and $y \in J'_k = f^{i_0}(J_k)$ for some k and some i_0. Define $\rho(I_i,\ J_j) = inf\{|s-y| : x \in J_i,\ y \in I_j\}$. Denote $\delta_k = min_{1 \le i \ne j < 2^k}\{\rho(I_i,\ I_j)\}$. Then $\delta > 0$. Note that

$$|f^i(x) - f^j(y)| \ge \rho(f^i(J_k),\ f^i(J'_k)) \ge \delta_k$$

for all $i \ge 0$. Thus $\liminf_{i\to\infty} |f^i(x) - f^i(y)| > 0$

Therefore ω contains only one point of a scrambled set.

Next we prove the second statement of the theorem. First, note that $\Omega \setminus \Lambda^2 = \Omega \setminus AP$ is countable. Also, the second condition of chaos in the sense of Li-Yorke implies that $\omega(x) \cap \omega(y) \ne \phi$ and two minimal sets are disjoint. If AP contains two points of a scrambled set, they must be in a minimal set. But a minimal set is an ω–limit set of some point in the minimal set. By the first part of the theorem, the two points are the same. Therefore Ω contains at most countably many points. ∎

Let $\mathcal{C}(\overline{R} = R)$ and $\mathcal{C}(\overline{P} = P)$ denote the sets of continuous mappings from I to itself with the properties $\overline{R} = R$ and $\overline{P} = P$ respectively. Let f be a continuous map. Denote $Orb_+(x) = \{y :\ y = f^n(x), n \le 0\}$ and $Orb_-(x) = \{y :\ f^n(y) = x, n \ge 1\}$.

In the following arguments, we assume that $f \in \mathcal{C}(2^\infty)$. Let $\overline{R}(f)^-$ denote periodic points and the points in $\overline{R} - P$ which are limit points of periodic points on their left side. Let $x \in \overline{R}(f)^- - R(f)$.

LEMMA 5.2.2. *Define* $p_{2^n} = max\{y < x :\ f^{2^n}(y) = y\}$. *There is an integer* $N > 0$, *such that for any* $z \in (p_{2^n},\ x)$, $f^{2^n}(z) > z$ *as* $n \ge N$.

PROOF: By Proposition E in [Xio, 1988] and $x \in \overline{R(f)}^-$, we know $f^{2^n}(x) > x$ as $n \ge N$ for some $N > 0$, since there is no periodic point with period less than 2^n on $(f^{2^n}(x);\ x)$.

If there exists a $z_0 \in (p_{2^n},\ x)$ such that $f^{2^n}(z_0) \le z_0$, then $f^{2^n}((z_0,\ x)) \supset (f^{2^n}(z_0), f^{2^n}(x)) \supset (z_0,\ x)$. Thus there exists a fixed point of f^{2^n}, a contradiction to the definition of P_{2^n}. ∎

LEMMA 5.2.3. *There exists an integer N such that $(p_{2^n},\ x) \cap Orb_+(x) = \phi$ as $n \geq N$ and $f^{2^m k}(x) \in (x,\ f^{2^n}(x))$ where $k = 1,\ 2,\ \ldots$ with $m > n \geq N$.*

PROOF: For the same reason as in the proof of Lemma 5.2.1, there exists an integer $N_1 > 0$, such that $f^{2^n}(x) > x$ as $n > N_1$. Since $x \notin \omega(x)$, there exists an integer $N_2 > 0$ such that $(p_{2^n},\ x) \cap Orb_+(x) = \phi$ as $n \geq N_2$.

Let $N = max\{N_1, N_2\}$. We can see that $(p_{2^n},\ x) \cap Orb_+(x) = \phi$ and $f^{2^m}(x) > x$ as $m > N$, $n \geq N$. As $m > n \geq N$, by Proposition E in [1], on $(x,\ f^{2^m k}(x))$, there exists no periodic point with period less than 2^m. Since there is a periodic point with period 2^n between $f^{2^n}(x)$ and $f^{2^{n+1}}(x)$, we have $f^{2^m k}(x) < f^{2^n}(x)$. Note that $f^{2^m k}(x) > p_{2^n}$ and $(p_{2^n},\ x) \cap Orb_+(x) = \phi$, we have $f^{2^m k}(x) > x$. Thus $f^{2^m k}(x) \in (x,\ f^{2^n}(x))$. ∎

We define $y \in \alpha_+(x)$ if for any $\epsilon > 0$, we have $(y,\ y+\epsilon) \cap Orb_+(x) \neq \phi$. Similarly we can define $\alpha_-(x)$.

LEMMA 5.2.4. *There exists an $\epsilon > 0$ such that for any points $p_1, p_2 \in (\overline{P} - \alpha(x)) \cap (x - \epsilon,\ x)$ with $p_1 < p_2$, if $(p_1,\ p_2) \cap Orb_-(x) \neq \phi$, then $(p_1,\ p_2) \cap \alpha(x) \neq \phi$. Also, there exist a point $y \in (p_1,\ p_2) \cap \alpha(x)$ and an integer m such that if $l \geq N$, $\{y_{l2^m}\}$ is monotonely increasing or decreasing. $y_{l2^m} \to y$ with $f^{l2^m}(y_{l2^m}) = x$ or x_{-1} where x_{-1} satisfies $f^{n_1}(x_{-1}) = x$ for some n_1.*

PROOF: By Proposition C in [Xio, 1988], $x \in \alpha(x)$. Then there exists a point x_{-1} near x such that $f^{n_1}(x_{-1}) = x$ for some $n_1 > 0$. Without loss of generality, suppse $(x_{-1},\ x) \cap Orb_+(x) = \phi$ since $x \in \overline{R^-} - R$.

First we suppose that $p_1, p_2 \in P$ and $f^{2^n}(p_1) = p_1$, $f^{2^n}(p_2) = p_2 (m > n)$ and $f^k(y) = x$ with $y \in (p_1,\ p_2)$. Then either $f^{l2^m}(y) = f^{l2^m - k}(x) > x$ or x_{-1} for any $l \geq N$ so that $l2^m - k > 0$. And $f^{l2^m}(p_1) = p_1$, $f^{l2^m}(p_2) = p_2$. So, $f^{l2^m}((p_1, y)) \supset [p_1, x]$ and $f^{l2^m}((p_2, y)) \supset [p_2, x]$. Then there are points $y^1_{l2^m} \in (p_1,\ y)$ $(y^2_{l2^m} \in (y,\ p_2))$

such that

$$(*) \qquad f^{l2^m}(y^1_{l2^m}) = f^{l2^m}(y^2_{l2^m}) = x \text{ (or } x_{-1}).$$

Since $x \notin R$, it is easy to see that $y^i_{l'2^m} \neq y^i_{l2^m}$ for $i = 1, 2$ with $l \neq l'$.

We claim that $\{y^1_{l2^m}\}$ is decreasing and $\{y^2_{l2^m}\}$ is increasing. Since $f^{l2^m}(y^1_{l2^m}) = x$ or x_{-1} and

$$\begin{aligned} f^{(l+1)2^m}(y^1_{l2^m}) &= f^{2^m}(f^{l2^m}(y^1_{l2^m})) \\ &= f^{2^m}(x) \text{ (or } f^{2^m}(x_{-1})) \\ &> x \text{ (or } < x_{-1}). \end{aligned}$$

We can substitute y by $y^1_{l2^m}$ and l by $l+1$ in $(*)$. Then we know that there exists a point $y^1_{(l+1)2^m} \in (p_1,\ y^1_{l2^m})$ such that

$$f^{(l+1)2^m}(y^1_{l2^m}) = x \text{ (or } x_{-1}).$$

$\{y^1_{l2^m}\}$ is a decreasing sequence. Similarly we can see that $\{y^2_{l2^m}\}$ is an increasing sequence.

If one or all of p_1 and p_2 are in $\overline{P} - P - \alpha(x)$, say p_1, then we can find a sequence $\{p^i_1\}$ in P which approaches p_1. On $(p^i_1,\ p_2)$ there are two monotone sequences $\{z^1_i\}$ and $\{z^2_i\}$ which have limit points y_1 and y_2 respectively in $(p^i_1,\ p_2] \cap \alpha(x)$. If y_1 or y_2 is in $(p_1,\ p_2)$, we are done. Otherwise, $p_2 \geq y_1$ and $y_2 \geq p_1$ since $y_1, y_2 \in (p^i_1,\ p_2]$. But $p_1,\ p_2 \notin \alpha(x)$. So $y_1,\ y_2 \in (p_1,\ P_2)$.

§5.3. The Structure of Continuous Type 2^∞ Unimodal Maps

In this section, we suppose f is of type 2^∞ with infinitely many periods.

LEMMA 5.3.1. *If $C \subset I^\infty$ is a component, then $f^i(C) \cap f^j(C) = \phi$ for all $i \neq j$.*

PROOF: By Lemma 5.1.1, $I^\infty \subset I^n = \bigcup_{\sigma_n} I_{\sigma_n}$ for all $n > 0$ and $C \subset I_{\sigma_n}$ for some σ_n. We know that $f^i(I_{\sigma_n}) \cap f^j(I_{\sigma_n}) = \phi$ for $0 \leq i \neq j < 2^n$ by Lemma 5.1.1. Then $f^i(C) \cap f^j(C) = \phi$ for $0 \leq i \neq j < 2^n$.

For any $i \neq j > 0$, there will always be an n such that 2^n is bigger than i and j. It follows that $f^i(C) \cap f^j(C) = \phi$ for each pair of positive integers $\{i, j\}$. ∎

LEMMA 5.3.2. *If $C \subset I^\infty$ is a nontrivial component, so is $f^n(C) \subset I^\infty$ for all $n > 0$.*

PROOF: By induction, $f^{n-1}(C)$ is a nontrivial component. Since f is strictly monotone on $f^{n-1}(C)$, $f^n(C)$ must be a nontrivial component. ∎

LEMMA 5.3.3. *Let $C \subset I^\infty$ be a nontrivial component. If a sequence $x_n \in f^n(C)$ has a subsequence x_{n_i} such that $\lim_{i\to\infty} x_{n_i} = x$ for some $x \in I$, then $\lim_{i\to\infty} y_{n_i} = x$ for any subsequence $y_{n_i} \in f^{n_i}(C)$.*

PROOF: The proof follows from Lemma 5.3.1. ∎

LEMMA 5.3.4. *C is a component of I^∞ if and only if $C = \bigcap_{n=1}^\infty I_{\sigma_n}$ for a sequence of permutation σ_n satisfying $\sigma_n = \sigma_{n-1} i_n, i_n = 0$, or 1, i.e. $I_{\sigma_{n+1}} \subset I_{\sigma_n}$.*

PROOF: $\bigcap_{n=1}^\infty I_{\sigma_n} (\subset I^\infty, \neq \phi)$ is closed since $I_{\sigma_{n+1}} \subset I_{\sigma_n} \subset I^n$ is closed for each n. $\bigcap_{n=1}^\infty I_{\sigma_n}$ is also connected. If $a, b \in \bigcap_{n=1}^\infty I_{\sigma_n}$, then $a, b \in I_{\sigma_n}$ and $[a, b] \subset I_{\sigma_n}$ for each n. So, $[a, b] \subset \bigcap_{n=1}^\infty I_{\sigma_n}$. Therefore, $\bigcap_{n=1}^\infty I_{\sigma_n}$ is a closed connected subset of I^∞. Suppose $\bigcap_{n=1}^\infty I_{\sigma_n} = [c, d]$. For each $x \in I^\infty \setminus [c, d]$, there is a σ_n such that $x \notin I_{\sigma_n}$, that is $x \in I_{\sigma_n'}$. Since $I_{\sigma_n} \cap I_{\sigma_n'} = \phi$ and $[c, d] \subset I_{\sigma_n}$, for any $y \in [c, d]$, $(x; y) \cap D^\infty \neq \phi$ where $(x; y) = (x, y)$ or (y, x). Thus $\bigcap_{n=1}^\infty I_{\sigma_n}$ is a component.

If C is a component of I^∞, then $C \subset I^\infty \subset I^n$ for each n. For fixed n, $C \subset I_{\sigma_n}$ for some unique σ_n since I_{σ_n}'s are disjoint. So there is a sequence $\{I_{\sigma_n}\}$ such that $C \subset \bigcap_{n=1}^\infty I_{\sigma_n}$. Because $\bigcap_{n=1}^\infty I_{\sigma_n}$ is a component, $C = \bigcap_{n=1}^\infty I_{\sigma_n}$.

If $\sigma_n \neq \sigma_{n-1} i_n$, then $I_{\sigma_n} \subset I_{\sigma_{n-1}'}$ is disjoint from $I_{\sigma_{n-1}}$ for some σ_{n-1}'. So $\bigcap_{n=1}^\infty I_{\sigma_n} = \phi$. Therefore, $\sigma_n = \sigma_{n-1} i_n$. ∎

LEMMA 5.3.5. *If $C = [f^M(t), b]$ (or $[a, f^M(t)]$) is a nontrivial component of I^∞ for some positive integer M, then $C \cap Orb_-(t) = \phi$ and $f^m(C)$ is a nontrivial component with one endpoint $f^{m+M}(t)$. Furthermore, f^m is strictly monotone on C for all $m > 0$.*

PROOF: Suppose $[f^M(t), b] \cap Orb_-(t) \neq \phi$. Then there exist an integer $m \geq 0$ and a point $y \in [f^M(t), b]$ such that $f^m(y) = t$. Then y is a turning point of $f^{2^n}|_{I_{\sigma_n}}$ for some σ_n with $2^n > max\{m, M\}$ by the iii) of 4) in Lemma 5.1.1. Thus $y \in I_{\sigma_n}$ and also $[f^M(t), b] \subset I_{\sigma_n}$.

If $y \in I_{\sigma_n 1}$, i.e. $f^{2^n}|_{I_{\sigma_n}}$ is a lower unimodal map. Then $[f^M(t), b] \subset I_{\sigma_n 1}$. But in fact $[f^M(t), b] \subset I_{\sigma_n 0}$ since $f^M(t)$ is an endpoint of the closed set $\bigcup_{\sigma_n} I_{\sigma_n}$ by the ii) of 4) in Lemma 5.1.1, and also $f^M(t)$ is a left endpoint, a contradiction.

If $y \in I_{\sigma_n 0}$, i.e. $f^{2^n}|_{I_{\sigma_n}}$ is an upper unimodal map, then $f^{2^{n+1}}|_{I_{\sigma_n 0}}$ is a lower unimodal map with turning point y_0 by 3) of Lemma 5.1.1. Thus $y \in I_{\sigma_n 01}$. This is the case in the last paragraph if we substitute n by $n + 1$. It can't happen.

For any $m > 0$, f^m is monotone on C for if there is a m_0 so that f^{m_0} has a turning point y on C, then $y \in Orb_-(t)$ which contradicts the previous part of the lemma.

Since f is strictly monotone on each of the intervials $[0, t]$ and $[t, 1]$, f^m is strictly monotone on C. ∎

LEMMA 5.3.6. *If $C = [f^M(t), b]$ (or $[a, f^M(t)]$) is a nontrivial component of I^∞ with one endpoint in $Orb_+(t)$, then $f^{m+M}(t) \notin \overline{P}$ for all $m \geq 0$.*

PROOF: Bt Lemma 5.3.5, we need only prove that $f^M(t) \notin \overline{P}$ since $f^{m+M}(t)$ is also an endpoint of $f^m(C)$. $f^M(t) \notin P$ since $P \cap I^\infty = \phi$ by 4) of Lemma 5.1.1. Thus $f^M(t) \in \overline{P} \setminus P$ and is a limit point of a sequence of periodic points with periods smaller than 2^n for some integer n with $2^n > M$. By 4) of Lemma 5.1.1 again, the periods of periodic points in D^n are finite. Thus $f^M(t) \in P$, a contradiction. ∎

LEMMA 5.3.7. *If $C \subset I^\infty$ is a component, then one endpoint of C is in R. $R \setminus P$ consists of trivial components of I^∞ and the endpoints of nontrivial components which are not in $Orb_+(t)$.*

PROOF: $C = \bigcap_{i=1}^{\infty} I_{\sigma_n}$ by Lemma 5.3.3. Then for each n, $C \subset I_{\sigma_n}$ and $f^{2^n}|_{I_{\sigma_n}} : I_{\sigma_n} \to I_{\sigma_n}$ is a unimodal map. For any given integer $N > 0$ and $\sigma_n = \sigma_N i_1 \dots i_{n-N}$ with $n > N$, we have $f^{2^n}(x) \in I_{\sigma_n}$ for all points $x \in C$ and $I_{\sigma_n} \subset I_{\sigma_N}$.

Let x_0 be an accumulation point of $\{f^{2^n}(x)\}$. Then $x_0 \in \omega(x)$ and $x_0 \in I_{\sigma_N}$. Since N is arbitrary, $x_0 \in C$. Note that $\Lambda = \overline{P} = R$. We have $x_0 \in \omega(x_0)$. By Lemma 5.3.1, x_0 can't be an interior point of C. The rest of this lemma is clear since $R \setminus P \subset I^{\infty}$. Therefore x_0 is an endpoint of C. ∎

In the proof of Lemma 5.3.7, we can see

REMARK. *$\{f^{2^n}(x)\}$ has at most two limit points. (Since $C \setminus int(C)$ contains only two points).*

LEMMA 5.3.8. *If C is a nontrivial component of I^{∞}, then either both endpoints of C are in $(R \setminus P) \cap \alpha(t) \cap \omega(t)$ or one of them is in $(R \setminus P) \cap \alpha(t) \cap \omega(t)$ and the other is in $Orb_+(t)$.*

PROOF: By Lemma 5.3.4, $C = \bigcap_{i=1}^{\infty} I_{\sigma_i}$. Since $I_{\sigma_i} \setminus int(I_{\sigma_i}) \subset Orb_+(t)$, $C \setminus int(C) \subset \overline{Orb_+(t)} = Orb_+(t) \cup \omega(t)$.

Suppose $C = [a, b]$. By Lemma 5.3.7, at lease one of a and b is in R. But by Lemma 5.3.6, $f^m(t) \notin \overline{P} = R$ for every potisive integer m. Either one of a and b is in $Orb(t)$ and the other is in $\omega(t)$ or both a and b are in $\omega(t)$.

Now we prove that $\omega(t) \subset (R \setminus P) \cap \alpha(t)$, so that $\omega(t) \subset (R \setminus P) \cap \alpha(t) \cap \omega(t)$.

Note that $\{f^3(t), f^4(t), ..., f^{2^{n+1}}(t)\}$ is the set of endpoints of $\overline{D} \cup \overline{D^1} \cup \dots \cup \overline{D^n}$. Let $a \in \omega(t)$. There exists σ_{k_i} such that $f^{n_i}(t) \in \overline{D_{\sigma_{k_i}}}$ and $f^{n_i}(t) \to a$. The for turning points $y_{\sigma_{k_i}} \in \{y \in I_{\sigma_{k_i}} : f^m(y) = t \text{ for some } m \geq 0\}$ of $f^{2^{k_i}}$, by Lemma 5.1.2, $y_{\sigma_{k_i}} \to a$. So $a \in R \cup \alpha(t)$.

Obviously $a \notin P$. Therefore $a \in (R \setminus P) \cap \alpha(t)$. ∎

REMARK. *If $C = [a, b]$ is a nontrivial component of I^{∞} with $a \neq b$ and a (or b) $\in (R \setminus P) \cap \alpha(t) \cap \omega(t)$, then a (or b) is approached on one side by its orbit.*

REMARK. $\omega(t) \subset (R \setminus P) \cap \alpha(t)$.

LEMMA 5.3.9. *If C is a nontrivial component of I^∞, then $Orb_-(t)$ contains at most one interior point of C.*

PROOF: Suppose that there are two points $y_1 \neq y_2$ and $f^{m_1}(y_1) = t$ and $f^{m_2}(y_2) = t$. If $m_1 \neq m_2$, then $f^{m_1}(C) \cap f^{m_2}(C)$ is not empty. This is a contradiction to Lemma 5.3.1.

If $m_1 = m_2$, for any n with $2^n > m_1$, y_1 and y_2 are turning points of f^{2^n} on some I_{σ_n}, a contradiction to the unimodality of f^{2^n} on I_{σ_n}

So $Orb_-(t) \cap int(C)$ is either empty or unique.

LEMMA 5.3.10. $R \setminus P = \overline{A} \setminus P \subset \alpha(t) \cap I^\infty \subset (\Omega \setminus P) \cap I^\infty$ *where* $A = \bigcup_{n=0}^{\infty} A_n = \bigcup_{n=0}^{\infty} \bigcup_{\sigma_n} \{\beta_{\sigma_n}\}$ *with* $\beta_{\sigma_0} = \beta$.

PROOF: We will prove this lemma in three steps.

1)We will show $R \setminus P \subset \overline{A} \setminus P$. Then $R \setminus P = \overline{A} \setminus P$.

Let $x \in R \setminus P$. There is a sequence p_i of periodic points with period m_i such that $m_i \to \infty$ and $\lim_{i \to \infty} p_i = x$. Since $p_i \in D^\infty$ for each i, there is a sequence of permutations σ_{m_i} so that $p_i \in D_{\sigma_{m_i}}$ for each i. Since $\beta_{\sigma_{m_i}} \in D_{\sigma_{m_i}}$, it follows from Lemma 5.1.2 that $\beta_{\sigma_{m_i}} \to x$. So, $x \in \overline{A}$. Then $x \in \overline{A} \setminus P$ since $x \notin P$.

2) $\overline{A} \setminus P \subset \alpha(t) \cap I^\infty$.

For any $x \in \overline{A} \setminus P$, there is a sequence of periodic points $\{\beta_{\sigma_{n_i}}\}$ with periods 2^{n_i} such that $\beta_{\sigma_{n_i}} \to x$. By (5′) of Lemma 5.1.1, it is easy to see that $x \in \alpha(t)$. Also $x \in I^\infty$ since $x \in \overline{A} \setminus P = R \setminus P \subset I^\infty$.

3) $\alpha(t) \cap I^\infty \subset (\Omega \setminus P) \cap I^\infty$.

Let $x \in \alpha(t) \cap I^\infty$. Let C denote the component of I^∞ with $x \in C$. By Lemma 5.3.10, $int(C) \cap \alpha(t) = \phi$, so $x \in C \setminus int(C)$. Thus, by Lemma 5.3.8, $x \in Orb(t) \cup R \setminus P$.

If $x \in R \setminus P$, then $x \in \Omega \setminus P$. So suppose that $x = f^m(t)$ for some positive integer m. Since $x \in \alpha(t)$, there is a sequence y_i of points in $Orb_-(t)$ with $y_i \to x$.

For each i, there is a positive integer m_i with $f^{m_i}(y_i) = t$. Then $f^{m_i+m}(y_i) = x$, so by Lemma 5.3.1.1, $x \in \Omega$.

Therefore $x \in \Omega \setminus P$ since $x \notin P$. ∎

LEMMA 5.3.11. *For any* $x \in I^\infty, \omega(x) = R \setminus P$.

PROOF: Since $\Lambda(f) = \overline{R(f)}$ by [Nit, 1980] and [Sar, 1982], and $\overline{R(f)} = R(f)$ by Lemma 5.1.3, $\omega(x) \subset \Lambda(f) = \overline{R(f)} = R(f)$. $\omega(x) \subset I^\infty$ since $f(I^\infty) \subset I^\infty$ and I^∞ is closed. Then $\omega(x) \subset \Lambda(f) \cap I^\infty = R \setminus P$. ∎

Now we prove that $y \in \omega(x)$ if $y \in R \setminus P$.

By Lemma 5.3.7, y is an endpoint of a component C (C may be trivial). By Lemma 5.3.3,

$$C = \bigcap_{n=1}^{\infty} I_{\sigma_n} \text{ and } x \in \bigcap_{n=1}^{\infty} I_{\sigma'_n}$$

where $\sigma_n = \sigma_{n-1} i_n, \sigma'_n = \sigma'_{n-1} i'_n$.

For each n, $x \in I_{\sigma'_n}$ and there exists m_n such that $f^{m_n}(x) \in I_{\sigma_n}$ by 2) of Lemma 5.1.1. Fix an integer n, then $f^{m_i}(x) \in I_{\sigma_n}$ for all $i \geq n$ since $I_{\sigma_{i+1}} \subset I_{\sigma_i}$.

Let x_0 denote an accumulation point of $\{f^{m_i}(x)\}$, then $x_0 \in I_{\sigma_n}$ since I_{σ_n} is closed. Since n is closen arbitrarily, $x_0 \in C$. Then $y \in \omega(y) = \omega(x_0) \subset \omega(x)$.

But $int(C) \cap \omega(x) = \phi$ by Lemma 5.3.7, so $x_0 \in C \setminus int(C)$. If $x_0 = y$, then $y \in \omega(x)$. If x_0 is another endpoint, then by Lemma 5.3.4, $y \in \omega(x_0) \subset \omega(x)$ since $y \in \omega(y)$.

Therefore $\omega(x) \supset R \setminus P$. ∎

LEMMA 5.3.12. *Let* $D^m = \bigcup_{\sigma_m} D_{\sigma_m}$. *For any* $x \in D^\infty$, *if* $f^N(x) \in D^M$ *for some positive integers* N *and* M, *then* $f^{n+N}(x) D^m$ *for* $0 \leq m < M$ *and* $n \geq 0$.

PROOF: Since $f(I^M) \subset I^M$ and $D^M \subset I^M$ by Lemma 5.1.1, we have that $f^{n+N}(x) \in I^M$. Also we know that $I^M \cap D^m = \phi$ for $0 \leq m < M$, so $f^{n+N}(x) \notin D^m$. ∎

LEMMA 5.3.13. *Assume* $f^n(x) \in D^\infty$ *for all* $n \geq 0$ *and* $f^{n_i}(x) \to z \in I^\infty$. *Let* $f^{n_i}(x) \in D_{\sigma_{m_i}}$. *Then* $m_i \to \infty$ *as* $i \to \infty$.

PROOF: The proof is by contradiction. Let $M = max\{m_i; i = 1, 2, \dots\} < \infty$ and

$$D' = \bigcup_{k=0}^{M} D^k.$$

Then $f^{n_i}(x) \in D'$, $z \in \overline{D'}$ and also $z \in \overline{D'} \setminus D'$ since $z \in I^\infty$ and $I^\infty \cap D' = \phi$. Then $z \in \overline{D_{\sigma'_k}} \setminus D_{\sigma'_k}$ for some σ'_k since $\{D_{\sigma'_k}\}_{k=0}^{M}$ is a collection of disjoint sets. So, $z = f^{m_0}(t)$ for some integer m_0 by 4) of Lemma 5.1.1 and $f^{n_i}(x) \in D_{\sigma'_k}$ as $n_i \geq N > 0$ by the continuity of f.

CLAIM. *Let $y = f^N(x) \in D_{\sigma'_k}$ for some $0 \leq k \leq M$. Then $f^{m2^k}(y) \in D_{\sigma'_k}$ for all $m > 0$. If $f^n(y) \in D_{\sigma'_k}$, then $n = m2^k$.*

By the given condition, $f^n(y) = f^{n+N}(x) \in D^\infty$. If $f^n(y) \in \bigcup_{i=k+1}^{\infty} D^i$, then $f^{n_i}(x) = f^{(n_i-n-N)+n}(y) \notin D^k$ as $n_i \geq n + N$ by Lemma 5.3.12. This is a contradiction to $f^{n_i}(x) \in D'_{\sigma_k} \subset D^k$. So $f^n(y) \in \cup_{i=0}^{k} D^i$. Since $y = f^N(x) \in D_{\sigma'_k} \subset D^k$, again by Lemma 5.3.12, $f^n(y) = f^{n+N}(x) \notin \bigcup_{i=0}^{k-1} D^i$. Therefore, $f^n(y) \in D^k$.

Since $f^{2^k}(D_{\sigma'_k}) \subset I_{\sigma'_k}$ and $D'_{\sigma_k} = I_{\sigma_k} \cap D^k$, $f^{2^k}(y) \in D_{\sigma'_k}$. Therefore, by induction, $f^{m2^k}(y) \in D_{\sigma'_k}$ for all $m > 0$.

Now we prove the latter part of the claim. Since $f^{i+(m-1)2^k}(D_{\sigma'_k} \subset f^i(I_{\sigma'_k})) \subset f^i(I_{\sigma'_k})$ and $f^i(I_{\sigma'_k} \cap I_{\sigma'_k} = \phi$ for $0 < i < 2^k$, we have

$$f^{(m-1)2^k+i}(y) \notin D_{\sigma'_k}$$

as $0 < i < 2^k$.

So, if $f^n(y) \in D_{\sigma'_k}$, then $n = m2^k$. This proves the claim.

Since $f^{n_i}(x) = f^{n_i-N}(f^N(x)) \in D_{\sigma'_k}$, then $n_i - N = m_i 2^k$ by the claim. Then

$$f^{m_i 2^k}(y) = f^{n_i-N}(y) = f^{n_i}(x) \to z$$

and $f^{m_i+1 2^k}(y) \to f^{2^k}(z)$ as well as $f^{m_i+2 2^k}(y) \to f^{2^{k+1}}(z)$.

By the claim, $f^{(m_i+1)2^k}(y)$ and $f^{(m_i+2)2^k}(y)$ are all in $D_{\sigma'_k}$. By 1) and 2) of Lemma 5.3. A, f is invariant on I^∞. Then $f^{2^k}(z)$, $f^{2^{k+1}}(z) \in I^\infty$. Also by the disscussion above, they are in the closure of $D_{\sigma'_k}$. So, $z, f^{2^k}(z), f^{2^{k+1}}(z)$ are all the endpoints of $D_{\sigma'_k}$. Therefore, either z or $f^{2^k}(z)$ is a periodic point, a contradiction to the fact that $I^\infty \cap P = \phi$. That implies that m_i goes to infinity as i goes to infinity.

∎

LEMMA 5.3.14. *For any $x \in D^\infty$, either $\omega(x) = R \setminus P$ or $\omega(x)$ is a periodic orbit.*

PROOF: Given any $x \in D^\infty$. If $f^{n_0}(x) \in I^\infty$, then $\omega(x) = \omega(f^{n_0}(x)) = R \setminus P$. We can assume that $f^n(x) \in D^\infty$ for all $n \geq 0$. If $z \in \omega(x)$ and $f^{n_i}(x) \to z$, we suppose that $f^{n_i}(x) \in D_{\sigma_{m_i}}$. Then by Lemma 5.3.13, $m_i \to \infty$ as $i \to \infty$.

For $m_i > 0$, by Lemma 5.3.12,

$$f^{n+n_i}(x) \notin D^m$$

for all $n \geq 0$ and $0 \leq m < m_i$. That is $\omega(x) \cap U_{m=0}^{m_i-1} D^m = \phi$. Let i go to infinity. Then we get $\omega(x) \cap D^\infty = \phi$. By 1) of Lemma 5.1.1, $\omega(x) \subset I^\infty$. Since $I^\infty \cap P = \phi$ by 1) and 4) of Lemma 5.1.1, $\omega(x) \subset I^\infty \setminus P$. Moreover, $\Lambda = \overline{R} = R$. Thus $\omega(x) \subset R \setminus P$.

By Lemma 5.3.11, $R \setminus P = \omega(z) \subset \omega(x)$. Then we get $\omega(x) = R \setminus P$.

The rest of the proof is obvious. ∎

STRUCTURE THEOREM. *Let f be a Type 2^∞ unimodal map with infinitely many different periods. Let t be the turning point. There is a closed invariant set I^∞ which satisfies the following properties:*

i) For any $x \in I^\infty$, we have $\omega(x) = R \setminus P$. For any $x \in D_\infty = I \setminus I^\infty$, either $\omega(x) = R \setminus P$ or $\omega(x)$ is a periodic orbit.

Also $R \setminus P$ is an infinite minimal set and it is the unique such set.

ii) I^∞ consists of three classes of components.

1) Some points in $R \setminus P$;

2) Countably many nontrivial closed intervals with endpoints in $\alpha(t) \cap (R \setminus P)$ *and with no interior points in* $Orb_-(t)$.

3) Countably many nontrivial closed intervals, denoted by C_p, $p \geq 0$, *with two endpoints in* $Orb_+(t)$ *and* $\alpha(t) \cap (R \setminus P)$ *respectively, and finitely many sets, denoted by* $C_p, 0 > p \geq M$ *for some integer* $M < 0$, *each of them consists of finitely many closed intervals with endpoints in* $\alpha(t) \cap (R \setminus P)$ *and interior containing a point in* $Orb_-(t)$.

iii) $f|_{I^\infty}$ *is semi-conjugate to the Adding Machine.* $f|_{R \setminus P}$ *is at most two to one semi-conjugate to the Adding Machine.*

iv) There is an integer $M > 0$ *such that* $\Omega = R \cup \{y; y = f^m(t), m \geq M\}$, *and also there is an integer* $M' > 0$ *such that* $CR = R \cup \{y; y = f^m(t), m \geq M'\}$.

PROOF: i) We have proved the first part in Lemma 5.3.11 and Lemma 5.3.14. The rest is obvious.

ii) By Lemma 5.3.8, the collection of nontrivial components of I^∞ can be divided into two classes $\mathcal{C}$ and $\mathcal{V}$ where $\mathcal{C} = \{C :$ one endpoint of C is in $Orb(t)$ and the other is in $\alpha(t) \cap \omega(t) \cap R \setminus P\}$ and $\mathcal{V} = \{V :$ both endpoints of V are in $\alpha(t) \cap \omega(t) \cap R \setminus P\}$.

By Lemma 5.3.5, $C \cap (Orb_-(t)) \setminus \{t\}) = \phi$. And by Lemma 5.3.10, $\mathcal{V} \cap Orb_-(t)$ contains at most one point. For $p < 0$, let C_p denote the union of the nontrivial elements of $\mathcal{V}$ containing a point in $\{x \in I; f^{|p|}(x) = t\}$, and for $p \geq 0$ let C_p denote the nontrivial element of $\mathcal{C}$ with an endpoint $f^p(t)$. By Lemma 5.3.1 and Lemma 5.3.2, we can see that $f(C_p) = C_{p+1}$.

The number of the remaining elements in $\mathcal{V}$ is either countable or zero by Lemma 5.3.1. If it is countable, we can denote them by $\{V_p; p = 1, 2, \dots\}$.

Since any trivial component of I^∞ is contained in R by Lemma 5.3.7, we can see that $I^\infty = (R \setminus P) \cap (U_{p=-M}^{\infty} C_p) \cup (U_{p=N}^{\infty} V_p)$ where $-\infty \leq M \leq \infty$, or $I^\infty = (R \setminus P) \cap (U_{p=M}^{\infty} C_p)$.

If $M \geq 1$, all nontrivial components have no points in $Orb_-(t)$ as interior points. Then f^n is a homeomorphism on each component for $n > 0$. We claim that f^n is homeomorphism on I^∞ for $n \geq 0$. It suffices to show that, if $x \in C$ and $y \in K$ with $C \neq K$, then $f^n(x) \neq f^n(y)$. By Lemma 5.3.4, $C = \bigcap_{i=1}^{\infty} I_{\sigma_i}$ and $K = \bigcap_{i=1}^{\infty} I_{\sigma_i'}$. Since $C \neq K$, there must be a positive integer N such that $I_{\sigma_i} \neq I_{\sigma_i'}$, for $i \geq N$. That implies that $f^n(x) \neq f^n(y)$ for $2^N > n \geq 0$ since both I_{σ_i} and $I_{\sigma_i'}$ are invariant under f^{2^N}. Hence f^n is a homeomorphism.

Therefore we have proved ii).

iii) For any $x \in I^\infty$ by Lemma 5.3.4, there is a unique point $\sigma = i_1 i_2 \dots i_n \cdots \in \Sigma_2$ such that $x \in I_{i_1 i_2 \dots i_n}$ for all $n \geq 1$. Then we can define a map $h : I^\infty \to \Sigma_2$ which maps x to $i_1 i_2 \dots i_n \dots$. It is easy to observe that h is onto.

Note that if $x \neq y$ and $h(x) = h(y)$, then $f(x) \neq f(y)$ and $h(f(x)) = h(f(y))$. Thus we can define a map $g : \Sigma_2 \to \Sigma_2$ by $g = hfh^{-1}$. We claim that, for an upper unimodal map f, g is the Adding Machine. Thus I^∞ is semiconjugate to the Adding Machine. Since each lower unimodal map f is conjugate to an upper unimodal map $1 - f(1-x)$, I^∞ must be semiconjugate to the Adding Machine for lower unimodal maps.

Assume that $i_1 = 1, \dots, i_{n-1} = 1$ and $i_n = 0$ for some $n \geq 1$ and $g(i_1 i_2 \dots i_n \dots) = j_1 j_2 \dots j_n \dots$. We need show that $j_1 = 0, \dots, j_{n-1} = 0$ and $j_n = 1$. Let $\sigma_{n-1} = i_1 i_2 \dots i_{n-1}$ and $\sigma_n = i_1 i_2 \dots i_n$. We take considering in $I_{\sigma_{n-1}}$ and I_{σ_n}. Since f is an upper unimodal map, f is decreasing on $I_{\sigma_{n-1}}$. Thus $f(I_{\sigma_{n-1}0}) = I_{00\dots01}$. Hence g is the Adding Machine.

iv) By the Theorem 4.2.3, $\Omega(f) \subset \Lambda \cup Orb_+(t) = R \cup Orb_+(t)$. Hence there is an integer $M > 0$ such that $\Omega(f) \subset R \cup \{y : \ y = f^i(t), i \geq M\}$ or $\Omega(f) = R(f)$.

Any point in $int(I^\infty)$ is not chain recurrent. For $x \in I \setminus I^\infty$, if $\omega(x)$ is a periodic orbit, $x \in CR$ by Lemma 5.1.4. Assume $\omega(x) = R \setminus P \subset I^\infty$. Let ϵ be one half of the length of a neighborhood of x whose image under f^{2^n} is contained

in I^∞. Then x cannot be chained to itself with an ϵ-chain. Thus $x \notin CR$. Therfore $CR(f) \subset R \cup Orb_+(t)$. Hence there is an integer $M' > 0$ such that $CR(f) = R \cup \{y : y = f^i(t), i \geq M'\}$ or $CR(f) = R(f)$. ∎

CHAPTER 6

CONCLUSION AND FURTHER RESEARCH

I plan to find analogues of some results known for continuous maps on the interval for homeomorphisms of the plane and of surfaces which have the plane as a covering space, e.g. the annulus and the torus.

1. In Chapter 2, I have introduced a new definition of chaos, ω-chaos. The property of being ω-chaotic is preserved under conjugacy. On the interval, I have proved that a continuous map is ω-chaotic if and only if it has positive topological entropy. But this property is not generally true. We can ask whether or not it is true for surface homeomorphisms. A possible answer to this question will depend on the understanding of several properties analogous to properties of maps on the interval. I will start with studying the relation between the topological entropy and the order of braid types and try to obtain an analogue of Misiurewicz's Theorem about the topological entropy and periodic points.

2. Transitivity plays an important role in the study of chaotic maps. On the interval, transitive maps have positive topological entropy. It is not known yet whether it is possible that there exists a closed invariant proper subset and a trasitive map so that the restriction of the map on the subset has the same entropy as the map has on the whole interval. The answer is likely to be no. I am going to find it out. Furthermore this question can also be asked more generally as follows. On a compact metric space, is it possible that there exists a closed invariant proper subset and a transitive homeomorphism with positive topological entropy so that the restriction of the map on the subset has the same entropy as the map has on the whole space?

The answer may be yes in general but no on some class of compact metric spaces. I will try to understand this problem.

3. In Chapter 3, I have studied the relation between the recurrence properties of a continuous map on a compact metric space and the recurrence properties of the shift map on the inverse limit space induced by the map. In a paper with Chen, we have proved that if the sole bonding map has the shadowing property, then so does its shift map on the inverse limit space. We can study similar problems like this. For example, we may study the relation between the Hausdorff dimension of the inverse limit space with a sole bonding map and the Hausdorff dimension of the base space for a certain class of bonding maps. Certainly the metric on the inverse limit spaces is very important. One has to choose proper metric. I will pay some attention to this problem.

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BIOGRAPHICAL SKETCH

Shihai Li was born in Xingyang County, Henan Province, of the People's Republic of China in November, 1963. He attended Lanzhou University in September, 1979, and graduated in July, 1983, with a BS degree. In September, 1983, he enrolled in graduate school at the University of Science and Technology of China and received an MS degree in December, 1985. He was then employed by Tsinghua University in Beijing and taught in the Department of Applied Mathematics until August, 1986. In October, 1986, he began studying in the Department of Mathematics of the University of Florida and finished in August, 1991, with a PhD degree.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Louis Block

Louis Block, Chairman
Professor of Mathematics

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C. K. Hsieh

ChungKuo Hsieh
Professor of Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

James Keesling

James Keesling
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Doctor of Philosophy.

Christopher Stark

Christopher Stark
Associate Professor of Mathematics

This thesis was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1991

Dean, Graduate School